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ENERGY METHODS IN SELF-ADJOINT EIGNEVALUE PROBLEMS

I. VARIATIONAL THEORY OF THE SPECTRUM

by

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I. INTRODUCTION

1. This is the first of two reports in which the purpose is to present, in as self-contained a manner as feasible, the theory, with applications, of energy methods in linear self-adjoint eigenvalue problems with a discrete spectrum. The present report deals with the theory of the eigenvalue spectrum in such systems, and the second report will deal with the theory and applications of the Ritz-Galerkin and related energy methods. Attention will be focused on problems characterized by ordinary differential equations of the form $M(y) = \lambda N(y)$, where M and N are real linear differential operators to be defined in further detail subsequently, $y = y(x)$, x is real, and λ is an eigenvalue. M and N, and the associated (homogeneous) boundary conditions, will be assumed such that the problem is self-adjoint and with a discrete eigenvalue spectrum. Such problems, despite their apparently restricted nature, still include a considerable variety of specific physical and engineering applications, as exemplified in the book of Collatz[1]. The prototype of such problems may be considered to be the free vibrations of a Bernoulli-Euler beam, or the buckling of a column. The present report includes the following items: (1) Variational, minimum, and maximum-minimum characterizations of the eigenvalues, and relevant implications of these characterizations. (2) Use in (1) of the familiar technique of Lagrange multipliers to establish the recursive minimum properties of the eigenvalues. (3) Theorems relating to the energy, in addition to the Rayleigh, quotient, and the role of natural boundary conditions in the general case. (4) Proof, by the use of minimizing sequences, of the existence of admissible minimizing functions for the general eigenvalue problem $M(y) = \lambda N(y)$. Such a proof is especially relevant for Ritz type of energy approximation methods. (5) Completeness and closedness theorems in both the

N-norm and M-norm. Self-contained proofs will be furnished throughout.

In view of the long and continual use of energy methods in many types of applications, it appears worthwhile to present the theory of such methods at least for the class of problems considered herein. Moreover, the unified variational type of approach taken here appears especially desirable for this purpose, since this is essentially the approach on which the energy methods can be most directly based, at least in the self-adjoint systems with discrete eigenvalue spectra considered here. As will be seen, the variational approach can be made quite elementary for this class of problems, in the sense that only concepts familiar from "advanced calculus" will be needed. Even such an elementary approach affords an opportunity, as will be seen, to touch, albeit slightly, on such fields of classical and modern interest as real variable theory, calculus of variations, approximation theory and functional analysis. Despite the existing rich literature on self-adjoint eigenvalue problems, there is no single reference which contains an elementary, essentially self-contained account, from a variational point of view, of all of the items listed in the preceding paragraph. Moreover, in order to keep the exposition simple, self-contained, but essentially complete, the authors have furnished their own proofs or modified existing ones in various instances in items (1) - (5).

In connection with the available pertinent literature, it is noted that the theory of linear self-adjoint eigenvalue problems has been treated in the following four ways: (a) Differential equations; (b) Integral equations; (c) Operators in a Hilbert space; (d) Variational principles. These approaches are, of course, not mutually exclusive, and indeed the difference in the approaches is sometimes mainly a matter of relative emphasis. For example, (b) is based on the well-established Hilbert-Schmidt theory

of homogeneous Fredholm integral equations with a symmetric kernel, while (a) is based essentially on the inverse operator of M, which in turn involves properties of the associated Green's function. The kernel in (b), however, is closely related to the Green's function. The approach in (c) is an abstract generalization of the integral operators in (b) to operators with a certain set of general properties. Even (d), which means here an approach based on extremal properties of the Rayleigh or related energy quotients, is often treated in conjunction with (a) - (c), especially in connection with existence theorems. The differential equation approach is given in Refs. 1-3. Ref. 1 also gives an elementary theory (i.e., without Green's functions, etc.) for the special case of a second-order differential equation with the boundary conditions $y(a) = y(b) = 0$, and indicates its extension to a particular fourth-order differential equation with certain boundary conditions. Ref. 3, in addition to the differential equation approach, gives a self-contained variational approach based on the theory of M. Morse; however, both of the approaches given in Ref. 3 are in rather abstract terms in a highly generalized setting, and the variational approach there appears rather different from that to be given here.

In approach (b) the differential equation is transformed into an integral equation. If the differential equation $M(y) = \lambda N(y)$ is self-adjoint, and $N(y)$ is of the one-term type (as defined in Ref. 1; e.g., $N(y) = \rho(x) y$) then the differential problem can be made equivalent to finding the eigenvalues of a homogeneous Fredholm integral equation with a symmetric kernel. An exposition of this integral equation problem is given, for example, in Refs. 4-6. A Hilbert space approach is shown in, e.g., Refs. 7-9. The variational theory for self-adjoint eigenvalue problems is given in Refs. 4 and 10 for Sturm-Liouville (second-order) differential equations; variational

aspects for Sturm-Liouville equations are also included in Refs. 11 and 12. For the more general system $M(y) = \lambda N(y)$, a variational approach is shown in Refs. 8 and 13. It is noteworthy that only Refs. 10 and 13 give a proof of existence of the relevant minimizing functions by variational means, i.e. by means of minimizing sequences. Thus, there appears to be no such type of proof available in the English-language literature. Even Mikhlin[8], who gives a comparatively extensive account of variational principles, refers to his Integral Equations text[5] for a proof of a compactness property which is assumed in Ref. 8 for the purpose of an existance proof. Collatz[1], for the general case $M(y) = \lambda N(y)$, proves the existence of an admissible minimizing function for the Rayleigh quotient by means of Green's functions.

It is remarked, finally, that the type of problem to be considered here is a generalization of Sturm-Liouville problems. It is however, still fairly restricted. For example, the literature (including some of the references cited above) now contains energy principles and methods for complex operators, non-self-adjoint and/or nonlinear problems, and partial differential equations. Nevertheless, or perhaps even because of this, it is worthwhile to present in a unified review a reasonably complete, yet elementary variational theory of energy methods of determining eigenvalues at least for the class of problems considered herein.

II. VARIATIONAL THEORY FOR THE RAYLEIGH QUOTIENT

2. Self-adjoint eigenvalue problems. As in Refs. 1, 2, 8 and 13, the problem will be considered in which it is desired to find all the eigenvalues of λ for which a non-trivial solution exists to the linear homogeneous differential equation

$$M(y) = \lambda N(y) \tag{1}$$

subject to the boundary conditions

$$B(y) = 0 \tag{2}$$

Here, $M(y)$ and $N(y)$ are assumed of the form

$$\left.\begin{aligned} M(y) &= \sum_{\nu=0}^{m} (-1)^{\nu} [f_{\nu}(x)\, y^{(\nu)}(x)]^{(\nu)} \\ N(y) &= \sum_{\nu=0}^{n} (-1)^{\nu} [g_{\nu}(x)\, y^{(\nu)}(x)]^{(\nu)} \end{aligned}\right\} \tag{3}$$

where $f_{\nu}(x)$ and $g_{\nu}(x)$ are given (real) ν-times differentiable functions in $[a, b]$. It is assumed that $m > n \geq 0$. The boundary conditions (2) may be supposed as 2m (independent) conditions, each of the form

$$\sum_{\nu=0}^{2m-1} (\alpha_{\nu}\, y^{(\nu)}(a) + \beta_{\nu}\, y^{(\nu)}(b)) = 0$$

where $[a, b]$ is the domain of the independent variable x, and α_{ν} and β_{ν} are given constants which will be here assumed independent of λ. The domain $[a, b]$ throughout this analysis will be assumed finite. For convenience, functions which satisfy all of the boundary conditions and have continuous derivatives through order 2m in $[a, b]$ will be called "comparison" functions (a term used by Collatz). It may be seen that the problem formulated here includes, for example, free bending and torsional vibrations, and buckling,

of Bernoulli-Euler beams, and is, in fact, a generalization of second-order Sturm-Liouville problems.

For functions u(x) and v(x), the following notation is introduced:

$$\left.\begin{aligned}
(u, v) &\equiv (v, u) \equiv \int_a^b u\, v\, dx \\
\Phi(u, v) &\equiv (u, M(v)) \equiv \int_a^b u\, M(v)\, dx \\
\psi(u, v) &\equiv (u, N(v)) \equiv \int_a^b u\, N(v)\, dx \\
\Phi(u) &\equiv \Phi(u, u) \equiv \int_a^b u\, M(u)\, dx \\
\psi(u) &\equiv \psi(u, u) \equiv \int_a^b u\, N(u)\, dx
\end{aligned}\right\} \qquad (4)$$

The quantity (u, v) is often called the "inner product" of u and v. The operators M and N are called positive definite if, respectively,

$$\Phi(u) > 0 \quad ; \quad \psi(u) > 0 \qquad (5a, b)$$

for all comparison functions u not identically zero. For any (square integrable) function u(x)

$$||u|| \equiv (u, u) \equiv \int_a^b u^2\, dx \qquad (6)$$

will be called the norm of u. More generally, if operators such as M and N are positive definite, then $\Phi(u)$ and $\psi(u)$ may be called respectively the norms of u "in the energy of M and N," or more briefly, the "M-norm" and "N-norm" of u; they are in that case denoted respectively by $||u||_M$ and $||u||_N$.

The eigenvalue problem (1) - (2) is called self-adjoint if

$$\Phi(u, v) = \Phi(v, u) \; ; \quad \psi(u, v) = \psi(v, u) \qquad (7)$$

for all comparison functions u and v. For given M and N in the form of Eqs. (3), it will be found by successive integrations by parts that the self-adjointness of the system will depend only on the boundary conditions (cf. Eq. (34d) below), and will be satisfied under a variety of boundary conditions commonly encountered in practice.

3. Orthogonality of the principal modes. If the eigenvalue problem (1)-(2) is self-adjoint, then for any two distinct eigenvalues λ_i and λ_j, the corresponding eigenfunctions y_i and y_j will satisfy the following orthogonality relations:

$$\psi(y_i, y_j) = 0 ; \qquad \Phi(y_i, y_j) = 0 \qquad (8a, b)$$

The proof follows by first writing $M(y_i) = \lambda_i N(y_i)$, $M(y_j) = \lambda_j N(y_j)$. Multiply the first equation by y_j and integrate over [a, b] ; multiply the second equation by y_i and integrate over [a, b]. Now subtract the two resulting equations. Then by virtue of the self-adjointness the left side vanishes, while the right side becomes $(\lambda_i - \lambda_j)\psi(y_i, y_j)$. Thus, Eq. (8a) follows. Eq. (8b) then also follows from either of the two equations which were originally subtracted from each other.

If to a single eigenvalue λ_i there correspond r linearly independent eigenfunctions, $y_{i1}, \ldots, y_{ir}$, λ_i is said to be of multiplicity r. In this case, any linear combination of the eigenfunctions will also be an eigenfunction corresponding to λ_i, and it is then readily possible to thus combine $y_{i1}, \ldots, y_{ir}$ (Gram-Schmidt procedure[1, 4]) to obtain r mutually orthogonal eigenfunctions corresponding to λ_i.

From the orthogonality relation (8a) it can be readily proved that if the eigenvalue problem (1)-(2) is self-adjoint, and N is positive definite, then all the eigenvalues must be real. For, if λ_i were a complex eigenvalue,

with corresponding eigenfunction y_i, then from the realness of the coefficients of Eqs. (1) and (2), the complex conjugate $\lambda_j = \bar{\lambda}_i$ would also be an eigenvalue, with the complex conjugate eigenfunction $\bar{y}_i$. Eq. (8a) would then imply $\psi(y_i, \bar{y}_i) = 0$. Putting $y_i = u + iv$, $\bar{y}_i = u - iv$ this relation becomes $\psi(u) + \psi(v) + i[\psi(v,u) - \psi(u,v)] = 0$. Since u and v must each be comparison functions, the imaginary part of the left side of the latter equation vanishes, while by virtue of Eq. (5b) the real part must be positive. Thus, assuming the existence of a complex eigenvalue leads to a contradiction.

4. Rayleigh quotient. For a function u(x), the Rayleigh quotient R(u) in connection with problem (1) is defined by

$$R(u) \equiv (u, Mu)/(u, Nu) \equiv \Phi(u)/\psi(u) \tag{9}$$

If y_i is an eigenfunction of Eq. (1), with eigenvalue λ_i, and $\psi(y_i) \neq 0$, it is easily seen, by multiplying both sides of Eq. (1) by y_i and integrating over [a, b], that

$$\lambda_i = R(y_i) = \Phi(y_i)/\psi(y_i) \tag{10}$$

It may be noted from Eq. (10) that if M and N are both positive definite, then all the eigenvalues will be positive.

5. Variational calculus. The few elements of the calculus of variations which will be used subsequently here, will now be introduced.

A functional, I(y), is a quantity whose value is determined by a function, y(x). An important example of a functional is

$$I(y) = \int_a^b F(x, y, y', \ldots, y^{(k)})\,dx \tag{11}$$

where a and b are given constants, and F is a given function. Clearly I becomes determined once the (k times differentiable) function y(x) in [a, b]

is specified. The quantities $\Phi(u)$, $\psi(u)$ and $R(u)$ defined by Eqs. (4) and (9) are important specific examples of functionals. We shall be concerned with "admissible" functions $y(x)$ for which $I(y)$ is "stationary". Quite generally, an "admissible" function is any function obeying certain conditions (usually certain types of continuity and differentiability properties, in addition to certain boundary conditions) specified in advance. To define what is meant by a stationary I for some admissible function $y(x)$, consider a second function $y(x) + \delta y$, where

$$\delta y(x) = \epsilon \eta(x) , \tag{12}$$

ϵ is a small parameter, and $\eta(x)$ is an arbitrary function, independent of ϵ, such that $y + \delta y$ remains admissible. δy is called a variation in y. Due to such a variation, F will be changed by

$$\begin{aligned}\Delta F &= F(x, y + \epsilon \eta(x), \ldots, y^{(k)} + \epsilon\, \eta^{(k)}(x)) - F(x, y, \ldots, y^{(k)}) \\ &= \epsilon\left(\frac{\partial F}{\partial y}\eta + \frac{\partial F}{\partial y'}\eta' + \ldots + \frac{\partial F}{\partial y^{(k)}}\eta^{(k)}\right) \qquad (13a) \\ &\quad + \text{ terms in higher power of } \epsilon ,\end{aligned}$$

assuming that for all x in $[a, b]$ F is expandable in a power series in ϵ with a non-zero radius of convergence. The (first) variation, δF, in F is now defined by the terms in the first power of ϵ in ΔF, i.e.,

$$\delta F = \epsilon \left(\frac{\partial F}{\partial y}\eta + \frac{\partial F}{\partial y'}\eta' + \ldots + \frac{\partial F}{\partial y^{(k)}}\eta^{(k)} \right) \tag{13b}$$

Similarly, the functional I will be changed by ΔI, and this change to first powers of ϵ is defined as the (first) variation, δI, in I. Thus, if the limits a and b in (11) remain fixed,

$$\delta I = \int_a^b (\delta F)dx \tag{14}$$

The functional I is called stationary at the function y(x) when $\delta I = 0$ for arbitrary (admissible) variations of the form (12), and hence for arbitrary (but admissible) $\eta(x)$. The function y(x) is then called an " extremal" of I.

There is an analogy between finding the minimum (or maximum) of a functional I for admissible functions y(x), and finding a relative minimum (maximum) of a differentiable function f(x) at an interior point of a region (x_1, x_2). In the latter case, if a relative minimum occurs at a point $x = x_o$, one considers the values $f(x_o + \epsilon)$ in the neighborhood of x_o. Expanding, $f(x_o + \epsilon) = f(x_o) + \epsilon f'(x_o) + (\epsilon^2/2!) f''(x_o + \xi\epsilon)$, $0 \le \xi \le 1$. If $f(x_o)$ is to be a minimum in the neighborhood of x_o, then $f(x_o + \epsilon) \ge f(x_o)$ for all sufficiently small $|\epsilon| \neq 0$. If $f'(x_o) \neq 0$, then it is clear that (with f" finite) for sufficiently small $|\epsilon|$, the minimum condition will be violated for at least some range of values of ϵ. Hence a necessary condition for a relative minimum at x_o is $f'(x_o) = 0$. It is noted that this manner of obtaining this familiar necessary condition is equivalent to expressing $\Delta f \equiv f(x_o + \epsilon) - f(x_o)$ to first powers of ϵ, and equating the result to zero, with $\epsilon \neq 0$; that is, it is equivalent to requiring $\delta f = 0$. Analogous considerations hold for finding the absolute minimum of a functional. Suppose y(x) is an admissible function for which I attains a minimum. Then $I(y + \epsilon\eta(x)) \ge I(y)$ for all ϵ and any arbitrary admissible function $\eta(x)$. If one expands $\Delta I \equiv I(y + \epsilon\eta(x)) - I(y)$ in powers of ϵ, it is seen that a necessary condition that $\Delta I \ge 0$ for (at least) sufficiently small $|\epsilon| \neq 0$ is that the coefficient of ϵ vanish; this, in turn, is the condition

$$\delta I = 0 \tag{15}$$

Thus, a necessary condition that I be a minimum for some admissible function y(x) is that its variation, δI, vanish for (first order) small arbitrary admissible variations δy; that is, I must be stationary at y(x).

In actual evaluations of δI, it is useful to note that in general, $d/dx\,(\delta y) = d/dx\,(\epsilon\,\eta(x)) = \epsilon\,\eta'(x)$. But also, $\delta(dy/dx) = [y(x) + \epsilon\,\eta(x)]' - y'(x) = \epsilon\,\eta'(x)$. Hence

$$d/dx\,(\delta y) = \delta(dy/dx), \qquad \text{or} \qquad d(\delta y) = \delta(dy) \tag{16}$$

To familiarize the reader with the variational process and the δ notation, suppose it is required to find a differentiable function $y(x)$ ($\not\equiv 0$) such that

$$y(0) = y(1) = 0 \tag{17a}$$

and such that

$$I \equiv I_1 - \lambda\, I_2 \equiv \int_0^1 (x+1)\, y'^{\,2}\, dx - \lambda \int_0^1 y^2\, dx \tag{17b}$$

be stationary, where λ is a constant. First, it is noted that $\delta I_1 = \int_0^1 (x+1)\cdot \delta(y'^{\,2})\, dx$, since $(x + 1)$ is a fixed function in the variation. Moreover, $\delta(y'^{\,2}) = 2y'\;\delta y'$, since, by definition, only "infinitesimally small" variations are considered, i.e., the change here in $y'^{\,2}$ is to be found only to first powers of ϵ. Thus, one finds $\delta I_1 = 2\int_0^1 (x+1)\, y'\;\delta y'\, dx$. Now integrate by parts, with $u = (x+1)\, y'$, $dv = \delta y'\, dx = \delta(dy) = d(\delta y)$; hence $v = \delta y$. Then one finds $\delta I_1 = 2(x+1)\, y'\;\delta y\,\Big]_0^1 - 2\int_0^1 [\,(x+1)\, y'\,]'\;\delta y\, dx$. Since y and $(y + \delta y)$ must satisfy conditions (17a), it follows that $\delta y = 0$ at $x = 0$ and at $x = 1$. Consequently, $\delta I_1 = -2\int_0^1 [\,(x+1)\, y'\,]'\;\delta y\, dx$. In addition, $\delta I_2 = \int_0^1 \delta\,(y^2)\, dx = 2\int_0^1 y\;\delta y\, dx$. Thus, one finally obtains

$$\delta I = 2\left\{ \int_0^1 [\,(x+1)\, y')' - \lambda\, y\,]\;\delta\, y\, dx \right\} \tag{17c}$$

In order that $\delta I = 0$ for arbitrary (admissible) $\delta y(x)$, it follows from (17c) that it is necessary and sufficient that

$$((x+1)\,y')' - \lambda y = 0 \tag{17d}$$

Thus, y must be a solution of the (here of Sturm-Liouville type) differential equation (17d) and satisfy the boundary conditions (17a). It is thus clear that the variational problem (17c), (17a) is equivalent to the differential-equation problem (17d), (17a); a solution of either problem is a solution of the other. In this example, λ will actually be an eigenvalue, since it would be required to find the value(s) of λ (if any) for which non-trivial solutions of (17d), (17a) exist.

The following type of variational problem with constraints will also be of importance here. Let I(y) be defined by Eq. (11), where the limits a, b are fixed. The functions y are now constrained not only to be admissible* (as before) but also to satisfy s ("isoperimetric") constraint conditions of the form:

$$G_j \equiv \int_a^b g_j\,(x, y, y', \ldots, y^{(k)})dx = c_j \qquad (j = 1, \ldots, s) \tag{18a}$$

where the g_j are given functions, and the c_j given constants. Among all such functions y, find that for which I(y) is a minimum. From (18a) the variations in y must satisfy the conditions

$$\delta\, G_j \equiv \int_a^b (\delta\, g_j)\, dx = 0 \qquad (j = 1, \ldots, s) \tag{18b}$$

As in the problem without the constraints (18a), a necessary condition on y(x) is still that it make I stationary, but now under variations which satisfy (18b). This problem is analogous to finding the minimum of a (differentiable) function $f(x_1, \ldots, x_p)$, where the x_i are subject to constraints of the form $h_i\,(x_1, \ldots, x_p) = c_i$, $i = 1, \ldots, s$; $(s < p)$. In particular, the

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* i.e. k-times differentiable and satisfying the boundary conditions.

problem (15), (18b) can be solved by introducing Lagrange multipliers $\kappa_1, \ldots, \kappa_s$, which for the isoperimetric constraints will be constants (as yet unknown), and requiring

$$\delta \, (I + \kappa_1 G_1 + \ldots + \kappa_s G_s) = 0 \tag{18c}$$

without constraint conditions (except for admissibility) on y. Suppose the general solution of (18c) for admissible y is obtained. This solution will depend on the constants $\kappa_1, \ldots, \kappa_s$. The latter are then determined so that the s constraint conditions (18a) hold. If such constants exist, the resulting y is then the desired solution. For, first of all, it satisfies (18a). Moreover (18c) is satisfied for arbitrary (admissible) variations δy, and hence it will also be satisfied for those particular δy which satisfy (18b). However, for such variations, (18c) implies that (15) will hold. Thus, y is a solution of the problem (15) under variations satisfying (18b). Conversely, any solution y, of the problem (15), (18b) must also be a solution of (18c) with the κ_i as determined above. For, such a y will satisfy (18c) under the constraint conditions (18b), for any values of the κ_i. Therefore, such a solution must be a subset of solutions of (18c) without any constraints on y. Thus, this solution must be included in the general solution of (18c) for any κ_i, and must hence be the solution of (18c) for those κ_i for which (18a) is satisfied.

6. Stationary properties of the eigenvalues. It will now be shown that if the problem (1)-(2) is self-adjoint, then any eigenfunction y_i, with corresponding eigenvalue $\lambda = \lambda_i$, is such that the quantity $(\Phi(y) - \lambda\, \psi(y))$ will be stationary at $y = y_i$ with respect to comparison functions y.

The proof can be carried out rather easily by first noting that since M and N are linear operators

$$\delta\, M(y) = M(\delta y) \quad , \qquad \delta\, N(y) = N(\delta y) \tag{19}$$

From Eqs. (19) and (7) it follows that

$$\int_a^b y\delta\, M(y)\, dx = \int_a^b y\, M(\delta y)\, dx = \int_a^b M(y)\, \delta y\, dx.$$

Consequently, from (4),

$$\delta\, \Phi(y) = \int_a^b y\delta\, M(y)\, dx + \int_a^b M(y)\, \delta y\, dx = 2\int_a^b M(y)\, \delta y\, dx \qquad (20a)$$

Similarly,

$$\delta\, \psi(y) = 2\int_a^b N(y)\, \delta y\, dx \qquad (20b)$$

Equations (19) and (20) hold for any comparison function y and (first-order small comparison) variations δy. Suppose now that y_i is an eigenfunction of (1), (2) with corresponding eigenvalue λ_i. Multiply Eq. (1), with $y = y_i$, $\lambda = \lambda_i$, by δy_i and integrate with respect to x over [a, b]. Then by virtue of Eqs. (20a, b) it is immediately seen that the variational equation $(1/2)\,(\delta\, \Phi(y_i) - \lambda_i\, \delta\, \psi(y_i)) = 0$ is obtained. With $\lambda = \lambda_i$ kept fixed in these variations, the variational condition can be written as:

$$\delta\, \Phi - \lambda\, \delta\, \psi = \delta\, (\Phi - \lambda\, \psi) = 0 \qquad (21)$$

This proves the theorem.

The converse of the above theorem also holds. That is, if a comparison function y and λ are such that the variational condition (21) holds for arbitrary comparison variations δy, then y and λ are an eigenfunction and corresponding eigenvalue of the problem (1), (2) when the latter is self-adjoint. This follows by noting that Eq. (21) in conjunction with Eqs. (20a, b) will imply

$$\int_a^b (M(y) - \lambda\, N(y))\, \delta y\, dx = 0 \qquad (22)$$

Since (except for being a comparison function) $\delta\, y(x)$ is arbitrary, this

implies Eq. (1), and the theorem is proven.

It is now noted that when $\psi(u) \neq 0$, Eq. (21) is equivalent to the condition

$$\delta R = 0 \tag{23}$$

where R is defined by Eq. (9) and the variations are with respect to comparison functions. This follows by simply noting that $\delta R = (1/\psi^2) \cdot [\psi(u)\, \delta\Phi - \Phi(u)\, \delta\psi] = (1/\psi)[\delta\Phi - R\, \delta\psi]$. The condition (21), with $\lambda \equiv R$, is hence equivalent to condition (23).

Thus, if the problem (1)-(2) is self-adjoint, the eigenfunctions are the comparison functions for which the Rayleigh quotient R assumes stationary values, and the corresponding eigenvalues λ will be the corresponding stationary values of R.

The above stationary property of the eigenvalues is already sufficient to indicate certain computational procedures by energy methods, such as Galerkin's method, to obtain at least approximately, the eigenvalues and eigenfunctions of (1)-(2). This will be discussed in a subsequent report. It should, however, be noted that these stationary properties do not as yet establish the <u>minimum</u> characterizations of the eigenvalues. The latter properties will now be established.

7. <u>Existence of a greatest lower bound of R</u>. It will now be assumed that in addition to being self-adjoint, the operators M and N are respectively positive semidefinite and positive definite, i. e.

$$\Phi(u) \geq 0\,, \qquad \psi(u) > 0 \tag{24a}$$

for any comparison function u not identically zero. It then follows that

$$R(u) \geq 0 \tag{24b}$$

Thus, the Rayleigh quotient R(u) has a lower bound in this case, and hence it must have a greatest lower bound, B (say). Clearly, $B \geq 0$.

8. Minimum characterization of the lowest eigenvalue. The following theorem can now be readily proven. Let the eigenvalue problem (1)-(2) be self-adjoint. Moreover, let the Rayleigh quotient R(u) for comparison functions u have a greatest lower bound B, as is the case, for example, when the inequalities (24a) hold. Finally, let there exist a comparison function $u = y_1$ for which $R(y_1) = B$ Then the lowest eigenvalue of (1)-(2) will be $\lambda = \lambda_1 = B$, with $y = y_1$ a corresponding eigenfunction.

The proof follows readily by observing that under the conditions of the theorem, R(u) attains a minimum when $u = y_1$, and hence the first variation, δR, in R vanishes for arbitrary comparison variations, δu, in y_1 (Sect. 5). It then follows from the stationary properties of the eigenvalues established in Sect. 6, that $u = y_1$ is an eigenfunction. From Eq. (10), the corresponding eigenvalue is $\lambda_1 = R(y_1)$. Since $R(y_1) = B$, $\lambda_1 = B$. The fact that B is the lowest eigenvalue follows from the fact that if λ_i is any eigenvalue, with eigenfunction y_i, then from Eq. (10), $\lambda_i = R(y_i)$. But $R(y_i) \geq B$. $\therefore$ $\lambda_i \geq B$.

If y_i is any eigenfunction of (1)-(2) then $C_i\, y_i$ is also an eigenfunction, where C_i is an arbitrary constant. When $\psi(y_i) \neq 0$ it is therefore possible to "normalize" the eigenfunctions by specifying, for example, that $\psi(y_i) = 1$. Under the conditions of the theorem just proven the lowest eigenvalue λ_1 can then be characterized as the minimum value of $\Phi(u)$ with respect to all comparison functions u for which

$$\psi(u) = 1 \tag{25}$$

9. Recursive minimum characterization of the higher eigenvalues.

The higher eigenvalues of (1)-(2) can be characterized by the following theorem. Let the conditions of the preceding theorem hold. In addition, suppose there exists a comparison function $u = u_k$ satisfying (25), orthogonal to the first $(k - 1)$ eigenfunctions $y_1, \ldots, y_{k-1}$, in the sense of

$$\psi(u, y_i) = 0 \qquad (i = 1, \ldots, k - 1) \qquad (26)$$

and minimizing the integral $\Phi(u)$ with respect to comparison functions u subject to the constraints (25) and (26). Then u_k is an eigenfunction, with the corresponding eigenvalue $\lambda_k = \Phi(u_k)$. λ_k is the next higher eigenvalue.

This theorem can be proven using the familiar technique of Lagrange multipliers (Sect. 5). Thus, introducing Lagrange multipliers $\kappa_k, \kappa_1, \ldots, \kappa_{k-1}$ the conditions of the theorem imply $\delta [\Phi(u) - \kappa_k \psi(u) - \kappa_1 \psi(u, y_1) - \ldots - \kappa_{k-1} \psi(u, y_{k-1})] = 0$ when $u = u_k$. Using Eqs. (20a, b) and (4) this is found to imply: $2(\delta u, [M(u) - \kappa_k N(u)]) - \kappa_1 \psi(\delta u, y_1) - \ldots - \kappa_{k-1} \psi(\delta u, y_{k-1}) = 0$, when $u = u_k$; hence, since δu is arbitrary

$$2[M(u_k) - \kappa_k N(u_k)] = \kappa_1 N(y_1) + \ldots + \kappa_{k-1} N(y_{k-1}) \qquad (27)$$

Multiply both sides of Eq. (27) by y_i $(i = 1, \ldots, k-1)$ and integrate over $[a, b]$. Then by virtue of Eqs. (8a)and (7) it is found that $\kappa_i \psi(y_i) = 2\Phi(u_k, y_i) - 2\kappa_k \psi(u_k, y_i)$. But since $M(y_i) = \lambda_i N(y_i)$, the preceding equation implies $\kappa_i \psi(y_i) = 2(\lambda_i - \kappa_k)\psi(u_k, y_i)$. From Eq. (26), and $\psi(y_i) > 0$, it then follows that $\kappa_i = 0$ $(i = 1, \ldots k-1)$. Hence Eq. (27) yields $M(u) = \kappa_k N(u_k)$. This proves that u_k is an eigenfunction, with corresponding eigenvalue $\lambda_k = \kappa_k$. Applying this result recursively to $\lambda_2, \lambda_3, \ldots$ it is seen that each $\lambda_{k+1} \geq \lambda_k (k = 1, 2, \ldots)$. For, $\lambda_k = \min \Phi(u)$ with u subject to (25) and to fewer of the orthogonality conditions in (26) than λ_{k+1}. Thus

the set of permissible functions for u_{k+1} is only a subset of those for u_k ; hence $\lambda_{k+1} \geq \lambda_k$.

It remains to prove that with the λ_k thus characterized recursively, there can be no eigenvalue between λ_k and λ_{k-1}. This will then show that the characterization of the eigenvalues given in the theorem of this section exhausts all the eigenvalues. It will further show that the eigenvalues form a discrete spectrum. Consider, then, an eigenvalue λ' greater than λ_{k-1}, with eigenfunction u' satisfying (25). Then u' must be different from $y_1, y_2, \ldots y_{k-1}$. Hence by Eq. (8a), u' is orthogonal to $y_1, \ldots, y_{k-1}$. Thus u' satisfies the same constraint conditions as u_k. But by Eq. (10), $\lambda' = \Phi(u')$. Hence from the present characterization of λ_k, $\lambda_k \leq \lambda'$. Thus, there is no eigenvalue between λ_{k-1} and λ_k.

The recursive minimization theorem implies that (when it holds) the set (spectrum) of eigenvalues will be infinite in number (as well as discrete). Finally, it is noted that the recursive minimization theorem can also be formulated without the normalization requirement (25) by replacing $\Phi(u)$ in the theorem by R(u).

10. Maximim-minimum theorem and some implications. The preceding recursive minimum property of λ_k depends on the lower eigenfunctions $y_1, \ldots, y_{k-1}$. It is possible, however, to characterize the k'th eigenvalue independently of the lower eigenfunctions, by means of a maximim-minimum theorem, first emphasized by Courant. This theorem can be stated as follows. Consider all comparison functions u which are orthogonal to each of (k-1) given functions $\varphi_1, \ldots, \varphi_{k-1}$, that is $(u, N(\varphi_i)) = 0$ $(i = 1, \ldots, k-1)$. Consider now the minimum, R_{min}, with respect to u, of the Rayleigh quotient (9). Then the k'th eigenvalue λ_k is the maximum value of R_{min} which can

be obtained by varying the functions $\varphi_1, \ldots, \varphi_{k-1}$.

This theorem can be proven by first choosing a function in the form $u_o = \sum_{i=1}^{k} a_i y_i$, where y_i is the i'th normalized eigenfunction. A set of a_i, not all zero, can be found such that $(u_o, N\varphi_j) = 0$, $j = 1, \ldots, k-1$. For, these conditions will lead to (k-1) linear homogeneous algebraic equations in k unknowns $(a_1, \ldots, a_k)$. Such a set of equations will always have a non-trivial solution (which need not be unique). From the linearity of M and N, and Eqs. (8a, b), one then finds $R(u_o) = \sum_{i=1}^{k} a_i^2 \, \Phi(y_i) / \sum_{i=1}^{k} a_i^2$. But from Eqs. (10) and (25), $\Phi(y_i) = \lambda_i$. Hence $R(u_o) = \sum_{i=1}^{k} a_i^2 \lambda_i / \sum_{i=1}^{k} a_i^2$. This is a weighted average, with the non-negative weights a_i^2, of $\lambda_1, \ldots, \lambda_k$. Hence $R(u_o) \leq \lambda_k$, and thus $R_{min} \leq \lambda_k$. If the functions $\varphi_1, \ldots, \varphi_{k-1}$ are now permitted to vary, it is known from the recursive minimum property of λ_k that $R_{min} = \lambda_k$ when $\varphi_i = y_i$. Thus, the maximum value of R_{min} obtained by varying $\varphi_1, \ldots, \varphi_{k-1}$ is λ_k.

It may be noted that the max-min theorem can be stated with the orthogonality conditions on the comparison function u in the form $(u, \varphi_i) = 0$ $(i = 1, \ldots, k-1)$. The proof would proceed exactly as above, except that in the final step it is noted that now $R_{min} = \lambda_k$ when $\varphi_i = N(y_i)$.

The following implication of the max-min theorem should be noted. Consider two linear self-adjoint systems S and S' characterized by Eqs. (1)-(3), and assume for both sufficient conditions for which the preceding minimum and max-min theorems will hold. Let the boundary conditions be the same for S and S', but let the coefficients in S' be such that $R(u) \leq R'(u)$ for all comparison functions u. Then $\lambda_k \leq \lambda'_k$ in each mode (k). (In actual applications of this principle the Dirichlet forms of $\Phi(y)$ and $\psi(y)$, defined in Eqs. (28) below, are often useful). This is proven by first noting that among comparison functions of a given set, $R_{min} \leq R'_{min}$.

For, let u be the function of the set for which R is a minimum, and let u' be the function of this set for which R' is a miminum; then $R_{min} = R(u) \leq R(u') \leq R'(u') = R'_{min}$. This immediately shows that $\lambda_1 \leq \lambda'_1$. Moreover, for the higher modes, let v by any comparison function orthogonal (with the operator N) to the first (k-1) eigenfunctions of S. Then $\lambda_k = \min_v R(v) \leq \min_v R'(v)$. But by the max-min theorem for S', $\min_v R'(v) \leq \lambda'_k$. Hence $\lambda_k \leq \lambda'_k$. One application of this theorem is furnished by the enclosure theorem of Collatz [Ref. 1, pp. 131-137], which is a generalization of that first given by Temple.

A second important general implication of the max-min theorem is the following. If two self-adjoint systems S and S' satisty, as above, conditions for the existence of the minima of R, and are the same except that S' is more constrained than S, meaning that the class of comparison funticns for S' is only a subset of that for S, then the eigenvalues of S' will be at least as large as the respective eigenvalues of S. Proof: Let u be any comparison function for S, and u' any comparison function for S', each orthogonal to the first (k-1) eigenfunctions of S. Then, since the class of u' is a subset of the class of u, $\min_u R(u) \leq \min_{u'} R(u')$. But $\lambda_k = \min_u R(u)$. Moreover, by the max-min theorem for S', $\lambda'_k \geq \min_{u'} R(u')$. Hence $\lambda'_k \geq \lambda_k$.

III. THE ENERGY QUOTIENT

11. Dirichlet forms. The various extremum theorems of Ch. II concern the Rayleigh quotient R(u), defined by Eq. (9), where u is a comparison function. There is, however, a related quotient, to be called the " energy quotient" , which also plays an important role both in the theory and in applications. Before introducing this quotient, it is observed that with M(y) and N(y) given as in Eqs. (3), successive integration by parts will yield the following forms for $\Phi(y)$ and $\psi(y)$:

$$\left.\begin{aligned} \Phi(y) &= \int_a^b \Big[\sum_{\nu=0}^{m} f_\nu \,(y^{(\nu)})^2 \Big]\, dx + M_0(y) \\ \psi(y) &= \int_a^b \Big[\sum_{\nu=0}^{n} g_\nu \,(y^{(\nu)})^2 \Big]\, dx + N_0(y) \end{aligned}\right\} \qquad (28)$$

where M_0 and N_0 are quadratic forms in the end-point values of y and its derivatives up to the (2m-1)th and (2n-1)th orders, respectively. Eqs. (28) are called Dirichlet forms of $\Phi(y)$ and $\psi(y)$, and hold for all (suitably differentiable) y.

If the boundary conditions (2) make the system self-adjoint, and if y is a comparison function, then it will be found that the boundary terms M_0 and N_0 can in general be written respectively as quadratic forms in the end-point values of y and its derivatives up to only the (m-1)th and (n-1)th orders. A detailed proof of this is given in Ref. 13. A comparatively simple, indirect type of proof can also be given, using the variational properties established in section 6. Thus, from Eq. (20a), for comparison functions y,

$$\int_a^b M(y)\, \delta y \, dx = \delta\, (\Phi(y)/2) \qquad (29)$$

However, with M(y) given by Eq. (3), the left side of Eq. (29) can also be evaluated by successive integration by parts (noting, e.g., Eq. (16)). This will yield the form:

$$\int_a^b M(y)\,\delta y\,dx = \delta\left[\,(1/2)\int_a^b \left(\sum_{\nu=0}^{m} f_\nu\,(y^{(\nu)})^2\right) dx\,\right] + \left[\sum_{\rho=0}^{m-1} A_\rho(y)\,\delta y^{(\rho)}\right]_a^b \tag{30}$$

where

$$A_\rho(y) = \sum_{i=\rho+1}^{m} (-1)^{\rho+i}\,(f_i\,y^{(i)})^{(i-1-\rho)} \tag{31}$$

Similar equations hold with M(y) replaced by N(y), and m, f_i and A_ρ replaced by n, g_i and B_ρ. Eqs. (30) and (31) hold for any suitably differentiable functions y. From Eq. (29) it follows, however, that if y is a comparison function, then the right side of Eq. (30) must be expressible as the exact variation of some functional; in fact, comparing Eqs. (29) and (30), and noting Eq. (28), it is seen that for comparison functions y,

$$2\left[\sum_{\rho=0}^{m-1} A_\rho(y)\;\delta\;y^{(\rho)}\right]_a^b = \delta\,M_0 \tag{32}$$

Since, however, $\delta y^{(\rho)}$ appears only for $\rho \leq m-1$ in the left side of Eq. (32), it follows that in any self-adjoint case M_0 (and hence also the A_ρ) must be expressible in terms of only the $y^{(\rho)}$, $\rho \leq m-1$, when account is taken of all the specific boundary conditions.

12. Natural, geometric and dynamical boundary conditions. For a given variational problem, an "admissible" function may be defined, quite generally, as any function among the permitted class of competing functions for that problem. An admissible function for the Rayleigh quotient in conjunction with the self-adjoint system (1)-(3), for example, would be any

comparison function (as previously defined). A "natural" boundary condition for a given variational problem may be defined, in general, as one which would always be automatically satisfied by a solution of the problem, although the admissible functions are not constrained to satisfy this condition in advance. A variational problem equivalent to a given differential system (such as (1)-(3))may often be modified to enlarge the class of admissible functions by modifying the functional to be stationary. This will be illustrated here, and more generally in the next section, by an important and well-known principle for the system (1)-(3).

In connection with the system (1)-(3), it will be convenient to define "geometric" and "dynamical" boundary conditions, a terminology due to Biezeno and Grammel[14]. Let the maximum number of boundary conditions in Eqs. (2) that can be formed to contain only y and its derivatives of order (m-1) or lower be k. Then these k conditions will be called the "geometric" boundary conditions. The remaining (2m-k) conditions will include derivatives of order m or higher, and will be called the "dynamical" boundary conditions. Geometrical and dynamical boundary conditions are also sometimes called respectively "essential" and "suppressible", or "remaining", after Kamke[13]. It will be shown in Section 13 that by introducing a modified quotient (the "energy quotient") in place of the Rayleigh quotient, a variational problem equivalent to (1)-(3) can be formulated in which the admissible functions need only satisfy the geometric boundary conditions. The dynamical boundary conditions will in this case be automatically satisfied by any solution of the problem, and will therefore be the natural boundary conditions of the problem. This has been shown for second-order (Sturm-Liouville) systems by Courant and Hilbert[4], Sagan[11], and many others. Biezeno and Grammel[14] have shown this for fourth-order (beam) systems, and an extension

to general self-adjoint systems of 2m'th order has been given by Kamke[13]. A simplified treatment for this general 2m'th order case will be given in Section 13.

To illustrate the statements in the preceding paragraphs, as well as the more general analysis to be given in the succeeding Section, consider a beam of length L governed by the equation

$$(p(x)\, y'')'' = \lambda\ \rho(x)\, y \tag{33a}$$

$(p(x),\ \rho(x) > 0)$. By successively integrating by parts, it is found that for any (suitably differentiable) function $u(x)$ and (small) arbitrary variations δu,

$$\int_0^L [\,(pu'')'' - \lambda \rho u\,]\, \delta u\, dx = \delta\,[\, \int_0^L (pu''^2/2)\, dx - \lambda \int_0^L (\rho u^2/2)\, dx\,]$$

$$+ (pu'')'\ \delta u\,\Big|_0^L - pu''\ \delta u'\,\Big|_0^L \tag{33b}$$

Suppose that the boundary conditions are:

$$y(0) = y'(0) = 0; \quad (py'')'(L) = \beta y(L)\,; \quad py''(L) = -\alpha y'(L)$$

(fixed-elastic ends) and consider now only functions u which satisfy the geometric boundary conditions $u(0) = u'(0) = 0$. Moreover, in view of the given dynamical boundary conditions, put $pu''_L = -\alpha\, u'_L$, $(pu'')'_L = \beta\, u_L$ in the boundary terms of (33b). Then from (33b), Eq. (33a) is thus seen heuristically to lead to the following variational problem:

$$\delta\,[\, \int_0^L pu''^2\, dx + \beta u_L^2 + \alpha\, u'^2_L - \lambda \int_0^L \rho\, u^2\, dx\,] = 0 \tag{33c}$$

where the admissible function u is any function with continuous derivatives

in [0, L] through order 4* which satisfies the given geometric boundary conditions $u(0) = u'(0) = 0$. It will now be shown that (33c) is indeed equivalent to the original problem.

From (33b) it follows that for admissible functions u, i.e. satisfying here $u(0) = u'(0) = 0$, Eq. (33c) implies

$$\int_0^L [\,(pu'')'' - \lambda\rho u\,]\delta u\,dx + [\beta u_L - (pu'')'_L\,]\,\delta u_L + [\alpha u'_L + pu''_L\,]\,\delta u'_L = 0 \tag{33d}$$

If $\delta u(x)$ is arbitrary in (33d)(except for being admissible) and δu_L, $\delta u'_L$ are arbitrary**, and independent of each other, then (33d) implies Eq. (33a) together with the dynamical boundary conditions $pu''_L = -\alpha u'_L$, $(pu'')'_L = \beta u_L$. The latter are thus the natural boundary conditions in the variational problem (33c). Condition (33c) is equivalent to

$$\delta\,\overline{R}\,(u) \equiv \delta\,\frac{\int_0^L pu''^2\,dx + \beta\,u_L^2 + \alpha\,u'^2_L}{\int_0^L \rho\,u^2\,dx} = 0 \tag{33e}$$

$\overline{R}(u)$ is the "energy quotient" for this problem, and (33e) shows that the original differential equation problem formulated here is equivalent to making $\overline{R}(u)$ stationary with respect to admissible functions u only required to satisfy the geometric boundary conditions.

The preceding result will now be generalized to the 2m'th order self-adjoint system (1)-(3).

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* Actually, as might be surmised from the form (33c), u need only have a continuous derivative in [a, b] through order m-1 = 1, and a piecewise continuous derivative of order m = 2. This will be proven in Chap. IV.

** The importance of this arbitrariness will be illustrated in an actual application in a subsequent report.

13. <u>Energy quotient.</u> Using the Dirichlet forms (28), the "energy quotient" $\overline{R}(u)$, for self-adjoint systems governed by (1)-(3), will be defined by

$$\overline{R}(u) = \overline{\Phi}(u)/\overline{\psi}(u) \tag{34a}$$

where

$$\overline{\Phi}(u) = \int_a^b \left[\sum_{\nu=0}^{m} f_\nu \left(u^{(\nu)}\right)^2 \right] dx + \overline{M}_0(u) \tag{34b}$$

$$\overline{\psi}(u) = \int_a^b \left[\sum_{\nu=0}^{n} g_\nu \left(u^{(\nu)}\right)^2 \right] dx + \overline{N}_0(u) \tag{34c}$$

and $\overline{M}_0(u)$ and $\overline{N}_0(u)$ are the expressions for $M_0(u)$ and $N_0(u)$ after the values of u and its derivatives at the end points, given by the boundary conditions, have been inserted, and all the derivatives of u at the end points of order m and higher have thus been eliminated from M_0 and N_0. Note that if u is a comparison function, then $\overline{\Phi}(u) = \Phi(u)$, $\overline{\psi}(u) = \psi(u)$, $\overline{R}(u) = R(u)$. For the example in Sect. 12, it will be found that $\overline{M}_0 = \beta u_L^2 + \alpha u_L'^2$, $\overline{N}_0 = 0$.

It will also be useful to define, analogously, the quantities $\overline{\Phi}(u, v)$ and $\overline{\psi}(u, v)$ for admissible functions u and v. For this purpose, consider first $\Phi(u, v)$, as defined by (4), in conjunction with Eq. (3). Integrating by parts, the following form is obtained:

$$\Phi(u, v) = \int_a^b \sum_{\nu=0}^{m} f_\nu u^{(\nu)} v^{(\nu)} dx + \left[M_0(u, v)\right]_a^b \tag{34d}$$

where $M_0(u, v)$ is a bilinear form in $u^{(r)}$, $v^{(s)}$, $0 \leq r \leq m-1$, $1 \leq s \leq 2m-1$. Suppose now that in the boundary term $\left[M_0(u, v)\right]_a^b$ the values of u, v and their derivatives according to the self-adjoint boundary conditions are inserted, thereby eliminating all derivatives above order (m-1). The result will be denoted by $\overline{M}_0(u, v)$; $\overline{\Phi}(u, v)$ is then defined by

$$\overline{\Phi}(u, v) = \int_a^b \left(\sum_{\nu=0}^{m} f_\nu \, u^{(\nu)} v^{(\nu)} \right) dx + \overline{M}_o(u, v) \tag{34e}$$

with a similar definition of $\overline{\Psi}(u, v)$, with f replaced by g, and $\overline{M}_o$ by $\overline{N}_o$. It should be noted that if u and v are admissible functions (satisfying only the geometric boundary conditions), and the boundary conditions are self-adjoint, then

$$\left.\begin{aligned} \overline{M}_o(u, v) &= \overline{M}_o(v, u) \, ; \quad \overline{N}_o(u, v) = \overline{N}_o(v, u) \\ \overline{\Phi}(u, v) &= \overline{\Phi}(v, u) \, ; \quad \overline{\Psi}(u, v) = \overline{\Psi}(v, u) \end{aligned}\right\} \tag{34f}$$

It may also be noted that from the definitions, $\overline{\Phi}(u) \equiv \overline{\Phi}(u, u)$; $\overline{\Psi}(u) \equiv \overline{\Psi}(u, u)$. Finally it is observed that if u and v are comparison functions, then all barred quantities will be identical with the corresponding unbarred quantities.

The following variational principle will now be established. When the problem (1)-(3) is self-adjoint, the eigenfunctions y will be those 2m-times differentiable* functions u in [a, b] satisfying the geometric boundary conditions, for which $\overline{R}(u)$ assumes stationary values; these values will be the corresponding eigenvalues λ. In this connection the quantity $P \equiv (1/2)(\overline{\Phi} - \lambda \overline{\Psi})$ may be called the generalized total potential, since the preceding principle is seen to be a generalization, in one dimension, of the familiar principle of the stationary total potential in elasticity. The principle implies that the dynamical boundary conditions will be automatically satisfied.

To establish this principle it is first noted that from Eqs. (30) and (34b, c), it follows that the statement $\delta P = 0$ is equivalent, for any (suitably differentiable) functions y, to

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* See first footnote of Section 12.

$$2 \int_a^b [\, M(y) - \lambda N(y) \,] \, \delta y \, dx + \delta \overline{M}_o - [\, 2 \sum_{\rho=0}^{m-1} A_\rho(y) \, \delta y^{(\rho)} \,]_a^b + \lambda \left\{ [\, 2 \sum_{\rho=0}^{n-1} B_\rho(y) \, \delta y^{(\rho)} \,]_a^b - \delta \overline{N}_o \right\} = 0 \tag{35}$$

Suppose now that in $M_o(y)$, the m'th and higher derivatives of y have been eliminated in accordance with the (self-adjoint) boundary conditions. Then the resulting form $\overline{M}_o$ for M_o becomes:

$$\overline{M}_o(y) = [\, \sum_{\rho=0}^{m-1} \overline{A}_\rho(y) \, y^{(\rho)} \,]_a^b \tag{36a}$$

where the $\overline{A}_\rho$ (y) (some or all of which may be zero) are certain linear functions, depending on the particular boundary conditions, of y and/ or its derivatives up to (m-1)th order. By virtue of the symmetrical properties of $\overline{\Phi}$ (u, v) and $\overline{M}_o$ (u, v) for admissible functions (cf. Eqs. (34f)) it follows now that an equation corresponding to (32) will hold for variations in $\overline{M}_o$ with respect to admissible functions, i. e.

$$\delta \overline{M}_o(y) = 2 \, [\, \sum_{\rho=0}^{m-1} \overline{A}_\rho(y) \, \delta y^{(\rho)} \,]_a^b \tag{36b}$$

A similar expression holds for $\delta \overline{N}_o(y)$, with $\overline{A}_\rho$ replaced by $\overline{B}_\rho$. Thus, Eq. (35) becomes:

$$\int_a^b [M(y) - \lambda N(y)] \, \delta y \, dx - [\, \sum_{\rho=0}^{m-1} \{ A_\rho(y) - \overline{A}_\rho(y) \} \, \delta y^{(\rho)} \,]_a^b + \lambda \, [\, \sum_{\rho=0}^{n-1} \{ B_\rho(y) - \overline{B}_\rho(y) \} \, \delta y^{(\rho)}]_a^b = 0 \tag{37a}$$

In the example of Sect. 12 the equation corresponding to (37a) is Eq. (33d), in which $B_{(\rho)} = \overline{B}_\rho = 0$ for all ρ, $(A_o)_a = (\overline{A}_o)_a = (A_1)_a = (\overline{A}_1)_a = 0$,

$(A_0)_b = (pu'')'_L$, $(\bar{A}_0)_b = \beta u_L$, $(A_1)_b = -pu''_L$, $(\bar{A}_1)_b = \alpha u'_L$. Since $\delta v(x)$ is arbitrary, Eq. (37a) is equivalent to

$$M(y) - \lambda N(y) = 0 \tag{37b}$$

and

$$\left[\sum_{\rho=0}^{m-1} \{ \bar{A}_\rho(y) - A_\rho(y) + \lambda [B_\rho(y) - \bar{B}_\rho(y)] \} \delta y^{(\rho)} \right]_a^b = 0 \tag{37c}$$

where $B_\rho = \bar{B}_\rho = 0$ for $\rho > n-1$. Eq. (37b) shows that the differential equation (1) will be satisfied by y. Moreover, as regards Eq. (37c), suppose first that all of the boundary conditions are dynamical. Then the $\delta y^{(\rho)}$ at the end points will all be independent of one another, and Eq. (37c) implies

$$\bar{A}_\rho(\alpha) - A_\rho(\alpha) + \lambda [B_\rho(\alpha) - \bar{B}_\rho(\alpha)] = 0 \tag{37d}$$

$$\alpha = a, b; \quad \rho = 0, 1, \ldots, m-1$$

Eqs. (37d) may be considered as a set of 2m linear equations in the 2m unknowns (contained in the A_ρ and B_ρ) $y^{(k)}(a)$, $y^{(k)}(b)$, $m \leq k \leq 2m-1$. One solution is clearly that in which the $y^{(k)}(\alpha)$ have the values given by the dynamical boundary conditions, since then $A_\rho(\alpha) = \bar{A}_\rho(\alpha)$ and $B_\rho(\alpha) = \bar{B}_\rho(\alpha)$. Moreover, this will be the only solution of Eqs. (37d), since as can be seen from the form of $A_\rho(y)$ and $B_\rho(y)$ in Eq. (3[illegible]) the determinant of the equations will not be zero. In case some of the boundary conditions, say ℓ of them, are geometric this argument may be modified by noting that there will now be ℓ (linear) relations among the $\delta y^{(\rho)}(\alpha)$, with $(2m-\ell)$ of the $\delta y^{(\rho)}(\alpha)$ which will be independent of one another. Hence Eqs. (37c) will now yield $(2m-\ell)$ linear equations in $(2m-\ell)$ derivatives of y of m'th and higher orders at the end points. Once again, these will be found to have as their unique solution the $(2m-\ell)$ dynamical boundary conditions of the problem.

Thus the variational condition $\delta P = 0$ with respect to 2m-times differentiable functions satisfying the geometric boundary conditions has been shown to imply satisfaction of Eq. (1), together with the dynamical, as well as geometric, boundary conditions. It follows that the extremal solutions y will be eigenfunctions of (1)-(3), with corresponding eigenvalues $\lambda = R(y) = \overline{R}(y)$. In a manner previously shown for the corresponding theorem on the Rayleigh quotient (cf. Eq. (23)), the condition $\delta P = 0$ is equivalent to $\delta \overline{R} = 0$. Moreover, in a manner analogous to that for R, the energy quotient $\overline{R}$ can be characterized by minimum and max-min properties. In particular, all of the minimum and max-min theorems which have been established in Chapter II for the Rayleigh quotient R remain valid when R is replaced by $\overline{R}$, [illegible] is replaced by [illegible] in the orthogonality constraint conditions for the higher modes, and the comparison functions need only satisfy the geometric boundary conditions.

IV. EXISTENCE OF MINIMIZING FUNCTIONS. INFINITE GROWTH OF THE EIGENVALUES

14. Null spectra; continuous spectra. The minimum, and max-min, theorems which have been proven here for the eigenvalues and eigenfunctions of self-adjoint problems are thus far all conditional in nature inasmuch as it was assumed that certain relevant minimizing functions exist. In this chapter it will be proven, under suitable sufficient conditions, that these do exist.

In actual buckling and structural vibration problems, conditions are usually automatically sufficient for the existence of minimizing functions. In such cases, as already seen in Sect. 9, the eigenvalues are infinite in number, but discrete (denumerable), with a lowest eigenvalue λ_1. Thus, to one accustomed to such types of problems, and to such a set (spectrum) of eigenvalues, the existence of minimizing functions, which in specific cases would be equivalent, e.g., to the existence of buckling or vibration mode-shapes, might at first appear obvious, and hardly in need of proof. It will be seen, however, that the existence of such functions, even under "ordinary" sufficient conditions, is mathematically far from obvious, and is indeed one of the fundamental and more difficult, but interesting, aspects of the theory. Moreover, there are important classes of problems (occurring, e.g., in physics) involving linear differential equations in which the spectrum of eigenvalues can be of a quite different nature from the discrete spectrum indicated above. This may occur when the differential equation is singular at one (or both) end points, or when the domain $[a, b]$ is infinite. In such cases, the spectrum of eigenvalues may still be of the "customary" type indicated above, but it may also be of a variety of different types, depending on the differential equation and on the boundary conditions (which now may

even be less in number than the order of the differential equation). For example, no eigenvalues at all may exist, or the spectrum may consist of a <u>continuous</u> set of numbers. In the first case, the spectrum may be called "null", and in the second case the spectrum is called "continuous". A simple example will suffice here to indicate these possibilities.

Consider the ("Euler-Cauchy") differential equation

$$(x^2 y')' = \lambda y \tag{38a}$$

with the domain [0, 1] for x. Eq. (38a) has a (regular) singular point $x = 0$, but is nevertheless of the type (1), (3). The general solution of this equation is

$$y = Ax^{n_1} + Bx^{n_2} \tag{38b}$$

where A and B are arbitrary constants, and n_1, n_2 are the roots* of the equation

$$n^2 + n - \lambda = 0 \tag{38c}$$

(a) Suppose the boundary conditions are:

$$y(0) = y(1) = 0 \tag{38d}$$

Then the system (38a), (38d) is not only of the type (1) - (3) but is also self-adjoint. It will be readily found, however, that there are <u>no</u> values of λ for which a non-trivial solution for y will exist.

(b) Let the boundary conditions now be only

$$y(0) = 0 \tag{38e}$$

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* If $n_2 = n_1$, then the general solution is $y = x^{n_1}(A + B \log x)$.

Then (considering, for simplicity, only the real eigenvalues of λ) it will be found that the set of real values of λ for a non-trivial solution is $\lambda > 0$. This is a continuous spectrum. (If (38e) is replaced by $y(0)$ = finite, then the set of real values for λ will be $\lambda \geq 0$).

The above illustration indicates the possibilities of a continuous spectrum as well as of no (non-trivial) solution at all (null spectrum). Thus, the existence of minimizing functions, etc., appearing in the theorems of the preceding sections cannot mathematically be taken for granted apriori, but must be proven, under suitable sufficient conditions.

The type of proof, based on minimizing sequences, to be given here will also be relevant in the proof, to be given in a subsequent report, of the convergence of the Ritz-Galerkin type of methods in all modes.

15. Auxiliary theorems on point sets and functions. In the proofs to follow a number of well-known theorems on infinite sets S of real numbers will be used. First it is recalled that a limit point, or point of accumulation, of S is defined as a point P such that in any neighborhood of P there exists an element of S distinct from P. P itself may or may not be an element of S. The following theorems are now noted.

(a) If S has a lower bound, then it has a greatest lower bound.

(b) If S has a greatest lower bound B, then B must have at least one of the following two properties: B is a member of S, or B is a limit point of S. (The proof follows almost immediately from the definitions*).

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* By definition of greatest lower bound, there will be at least one element s in S such that $s < B + \epsilon$ for any $\epsilon > 0$. If B is not in S, then this element s must be distinct from B. Hence if B is not in S, B must be a limit point of S. (This does not preclude the possibility that B is both in S and also a limit point of S).

(c) If S has a limit point P, then there is a sequence of elements e_n in S such that $\lim_{n \to \infty} e_n = P$.

(d) If S is a bounded infinite set, then there exists at least one limit point of S. (Bolzano-Weierstrass theorem).

(e) An infinite sequence $\{s_k\}$ of non-decreasing real numbers with an upper bound is convergent; its limit is the least upper bound of the numbers s_k.

(f) An infinite sequence $\{s_k\}$ of non-increasing real numbers with a lower bound is convergent; its limit is the greatest lower bound of s_k.

Theorems (a) and (c)-(f) are proven, for example, in Ref. 15. In view of theorem (c), it is noted that the Bolzano-Weierstrass theorem (d) can also be stated as follows:

(g) If S is a bounded infinite set then there exists a sequence of elements e_n in S which converges to a limit point of S.

In addition to the above theorems on sets and sequences of numbers, several general theorems on functions will be needed. The notation of Eqs. (4) will be used.

(h) Let M be any linear differential operator, and let u, v be functions of a class for which the M-norm exists and for which M is symmetric, i.e., $\Phi(u, v) = \Phi(v, u)$ for all u, v of the class. Then

$$\Phi(ru + sv) = r^2\, \Phi(u) + \quad s\, \Phi(u, v) + s^2\, \Phi(v) \tag{39}$$

where r and s are any constants. Eq. (39) follows readily from the definitions. Suppose now that M is positive semidefinite. Then

$$(\Phi(u, v))^2 \leq \Phi(u) \cdot \Phi(v) \tag{40a}$$

The inequality (40a) follows by letting s = 1 in (39) and considering the right side of (39) as a quadratic in r. Since the latter cannot be negative for any

real r, it must either have no real roots or a double root. This requirement immediately yields (40a). For the special case of M = I, the identity operator, (40a) is the well-known Cauchy-Schwarz-Bunyakowsky inequality:

$$(u, v)^2 \leq (u, u) \cdot (v, v) \tag{40b}$$

(i) Let $f_j(x)$ be a sequence of differentiable functions in [a, b], and let the sequence $f_j'(x)$ converge uniformly in [a, b] to a function $\varphi(x)$. Moreover, let there be at least one point ξ in [a, b] for which the sequence $f_j(\xi)$ converges. Then the sequence $f_j(x)$ will converge uniformly in [a, b] to a function F(x), such that $F'(x) = \varphi(x)$. This is proven in Ref. 16(with series instead of sequences), and is related to differentiation of an infinite series or sequence.

(j) If for a sequence of functions $u_h(x)$, $I_h \equiv \int_a^b f(x)\, u_h^2(x)\, dx$ is bounded, and f(x) is continuous and > 0 in [a, b], then $J_h \equiv \int_a^b u_h^2(x)\, dx$ will be bounded. This is readily proven by noting that the hypotheses imply $I_h \leq A$, and also $f(x) \geq C > 0$ in [a, b],where A and C are fixed numbers. Hence $0 \leq J_h \leq (1/C) \int_a^b f(x)\, u_h^2\, dx \leq A/C$.

(k) Let $\varphi(x)$ be a given piecewise continuous function in [a, b], and w(x) an arbitrary function with a continuous (m+ 1)th derivative in [a, b] satisfying the boundary conditions

$$w^{(\nu)}(a) = w^{(\nu)}(b) = 0, \qquad \nu = 0, 1, \ldots, m \tag{41a}$$

Suppose that

$$\int_a^b \varphi\, w^{(m+1)}\, dx = 0 \tag{41b}$$

for all w. Then $\varphi(x)$ must be a polynomial, P_m, of m'th or lower degree. This is a theorem of Zermelo.

Proof. If $\varphi(x) = P_m$, then (41b) is satisfied, since

$$\int_a^b P_m(x)\, w^{(m+1)}(x)\, dx = 0 \tag{41c}$$

by successive integrations by parts, with the conditions (41a). To show that $\varphi(x)$ cannot be any other function, let $P_m(x) = C_o + C_1 x + \ldots + C_m x^m$, where the C_i are determined so that

$$\int_a^b x^i P_m(x)\, dx = \int_a^b x^i \varphi(x)\, dx, \qquad i = 0, 1, \ldots, m \tag{41d}$$

Eqs. (41d) are a set of (m + 1) linear equations in the (m + 1) unknowns $C_o, \ldots, C_m$, and will have a unique solution for a given $\varphi(x)$. For, in the special case in which $\varphi(x)$ is a polynomial of m' th or lower degree, the C_i would have to satisfy (41d), while for such a $\varphi(x)$ not identically zero, the right side would be non-zero at least for some i. Hence, since a solution for the C_i exists in that case, the determinant of the system in that case cannot vanish, and therefore cannot vanish for any given φ, since the determinant is the same for all φ. It is now noted from Eqs. (41b) and (41c) that for any given φ,

$$\int_a^b (\varphi - P_m)\, w^{(m+1)}\, dx = 0 \tag{41e}$$

for all w. Moreover, a w(x) exists (i.e., satisfying (41a)) such that

$$w^{(m+1)}(x) = \varphi(x) - P_m(x) \tag{41f}$$

The required w(x) is obtained by successive quadratures of (41f) each satisfying the boundary conditions at x = a in (41a). The conditions at x = b will then be automatically satisfied. For, the first quadrature yields $w^{(m)}(x) = \int_a^x [\varphi(x) - P_m(x)]\, dx$; by virtue of (41d) for i = 0, (41a) is satisfied for $\nu = m$. It is next noted that $I_1 \equiv \int_a^b x\, w^{(m+1)}(x)\, dx = x\, w^{(m)}\Big|_a^b$

$- \int_a^b w^{(m)} dx = - \int_a^b w^{(m)}(x)\, dx$. But (41d) with i = 1 and (41f) imply $I_1 = 0$. Hence $w^{(m-1)}(x) = \int_a^x w^{(m)}(x)\, dx$ will now satisfy (41a) for ν = m and m-1. By considering $I_2 \equiv \int_a^b x^2 w^{(m+1)}(x)\, dx$, etc. one can finally show that w(x) will satisfy all the boundary conditions of (41a). In (41e) now, let $w^{(m+1)}(x)$ be given by (41f). Then this will imply $\varphi(x) = P_m(x)$.

(l) Let $u_h(x)$ be an infinite sequence of functions each of which has continuous derivatives in [a, b] through some order p. Moreover, for some q, ($0 \leq q \leq p$), let there be an interval (x_1, x_2) in [a, b] for which the sequence $\int_{x_1}^{x_2} [u_h^{(q)}(x)]^2\, dx$ is bounded. Then there will exist a sequence of points $\xi_{h,\nu}$ in [a, b] such that for each ν, $q \leq \nu \leq p$, the sequence $u_h^{(\nu)}(\xi_{h,\nu})$ is bounded. This is an important theorem here which can be proven in the manner of Ref. 13, pp. 85-87, although the theorem is not explicitly stated there in this generalized form.

(m) Finally, the theorem of Arzela, also known as the theorem of Ascoli, must be noted. This is an extension of theorem (g) to functions. In stating this theorem it is first necessary to define a set of equicontinuous functions. A set of functions f(x) is said to be equicontinuous if for every $\epsilon > 0$ there exists a $\delta > 0$, depending on ϵ but not on the particular function f(x) of the set, such that if $|x_1 - x_2| < \delta(\epsilon)$, then $|f(x_1) - f(x_2)| < \epsilon$, where x_1 and x_2 are in the domain D of the independent variable. The theorem of Arzela states that a set of uniformly bounded and equicontinuous functions in a given bounded domain D is compact, i.e. from such a set it is possible to choose a sequence of functions which converges uniformly to a continuous limit function in D. This theorem is proven in Refs. 4 and 6.

16. <u>Existence of a minimizing sequence for $\overline{R}$.</u> An "admissible" function u for the energy quotient $\overline{R}(u)$ will henceforth be defined as one which satisfies the geometric boundary conditions, and has a continuous

(m-1)th derivative, and a piecewise continuous m'th derivative in [a, b]. It will be assumed that M is positive semidefinite and N positive definite. Then for any admissible function u, $\overline{\Phi}(u) \geq 0$, $\overline{\Psi}(u) > 0$ and hence $\overline{R}(u) \geq 0$. Therefore by auxiliary theorem (a), $\overline{R}(u)$ has a greatest lower bound, say B. Consider now the set S of all values $\overline{R}(u)$ for admissible functions u. If S is finite then the function for which $\overline{R}(u)$ assumes the lowest value will be the minimizing function, and the existence of a minimizing function for $\overline{R}$ is thus proven. Suppose, then, that S is infinite. Then from auxiliary theorem (b), B is a member of S or it is a limit point of S. In the former case, the existence of a minimizing admissible function for $\overline{R}$ is again proven. Hence suppose that B is a limit point of S. Then from auxiliary theorem (c), there exists a sequence of values of $\overline{R}(u)$, with the corresponding sequence $\{u_h\}$ of admissible functions u_h, such that

$$\lim_{h \to \infty} \overline{R}(u_h) = B \tag{42}$$

A sequence such as u_h is called a minimizing sequence for $\overline{R}(u)$. Note that the existence of such a sequence has been proven here without the necessity of specifying how such a sequence is to be constructed in practice. (The actual construction of such sequences forms, in fact, the basis of such methods as the Rayleigh-Ritz to be discussed in a second report.)

The existence, just proven, of a minimizing sequence satisfying an equation of the type (42), does not per se necessarily imply that an admissible function U exists for which $\overline{R}(U) = B$. A simple illustration of this, due to Weierstrass and given in Ref. 8, is the following. Find a continuous function y, with a continuous derivative in [-1, 1], which satisfies the boundary conditions

$$y(-1) = -1, \qquad y(1) = 1 \tag{43a}$$

and for which the functional

$$I(y) \equiv \int_{-1}^{1} x^2 \, y'^2 \, dx \tag{43b}$$

is a minimum. Since $I(y) \geq 0$, $I(y)$ has a lower bound, and hence a greatest lower bound, say B, ≥ 0. The value of B is here zero, as can be shown by considering the set of functions $y_\epsilon(x)$ defined by

$$y_\epsilon(x) = \arctan(x/\epsilon) \big/ \arctan(1/\epsilon) \tag{43c}$$

where ϵ is a positive number. These functions are clearly admissible here. (They satisfy (43a) and have a continuous derivative in $[-1, 1]$).

Moreover,

$$I(y_\epsilon) < \int_{-1}^{1} (x^2 + \epsilon^2) \, y_\epsilon'^2 \, dx = 2\epsilon / \arctan(1/\epsilon) \tag{43d}$$

From (43d) it follows that $I(y_\epsilon)$ can be made as close to zero as desired by choosing a sufficiently small $\epsilon > 0$. Hence the greatest lower bound of $I(y)$ is $B = 0$, and a minimizing sequence for $I(y)$ would be $y_\epsilon(x)$, with (for example) $\epsilon = 1/n$, $n = 1, 2, \ldots$. Nevertheless there does not exist any admissible function y for which $I(y) = b = 0$. For, such a function would here have to be such that the integrand $x^2 \, y'^2$ is identically zero; the only such function is y = const., which cannot satisfy the boundary conditions (43a). It may also be noted, incidentally, that although each function $y_\epsilon(x)$ of the minimizing sequence (43d) is continuous in $[-1, 1]$, the limit $Y(x)$ of this sequence as $\epsilon \to 0$ is a discontinuous function: $Y(x) = -1$ for $-1 \leq x < 0$; $Y(0) = 0$; $Y(x) = 1$ for $0 < x \leq 1$. (The convergence of $y_\epsilon(x)$ to $Y(x)$ is non-uniform).

It will now be shown, under certain suitable sufficient conditions, that an admissible function U for which $\bar{R}(U) = B$ does exist. The proof, based on applying the theorem of Arzela to a minimizing sequence for $\bar{R}(u)$, will be along the lines of Ref. 13.

17. Assumptions. The problem defined by Eqs. (1) - (3) is assumed self-adjoint, and $f_\nu(x)$ and $g_\nu(x)$ are assumed to have continuous derivatives in [a, b] through orc. Moreover, it is assumed that:

$$\left.\begin{array}{l} f_m(x) > 0 \;; \quad f_\nu(x) \geq 0, \quad g_\nu(x) \geq 0 \quad (\nu = 0, \ldots, m-1) \\ f_o(x) + g_o(x) \not\equiv 0 \;; \quad \overline{M}_o(u) \geq 0 \;; \quad \overline{N}_o(u) \geq 0 \\ \overline{\Psi}(u) > 0 \quad , \quad \overline{\Phi}(u) \geq 0 \end{array}\right\} \quad (44)$$

where u is any admissible function not identically zero. These assumptions have all been grouped together for convenience. They are, of course, not entirely independent of one another. In particular, the assumptions on $f_\nu(x)$, $g_\nu(x)$, $\overline{M}_o(u)$ and $\overline{N}_o(u)$ imply $\overline{\Psi}(u) \geq 0$, $\Phi(u) \geq 0$. Here it is assumed in particular that $\overline{\Psi}(u) > 0$, i.e, N is a positive definite operator. The condition $f_o(x) + g_o(x) \not\equiv 0$ can be relaxed, as will be indicated subsequently. Although the assumptions (44) may perhaps at first sight appear somewhat restrictive, it is noted that, among other possible physical situations, they are typically satisfied in a variety of actual structural buckling and vibration problems.

From $\overline{\Psi}(u) > 0$, $\overline{\Phi}(u) \geq 0$ it follows that $\overline{R}(u) \geq 0$, and therefore that $\overline{R}(u)$ has a greatest lower bound, B. As shown in Section 16, there will hence exist a minimizing sequence of admissible functions $u_h(x)$, $h = 1, 2, \ldots,$ for $\overline{R}(u)$. Since $\overline{\Psi}(u_h) > 0$, the functions $u_h(x)$ may be normalized so that

$$\overline{\Psi}(u_h) = 1 \quad (45a)$$

Then from Eq. (42),

$$\lim_{h \to \infty} \overline{\Phi}(u_h) = B \quad (45b)$$

The nature and convergence of the sequence $u_h(x)$ and of the sequences of derivatives of $u_h(x)$ will now be analyzed.

18. Uniform convergence of $u_h^{(\nu)}(x)$, $\nu = 0, 1, \ldots, m-1$. The following statement will first be proved. (1) For every $0 \leq \nu \leq m-1$ and every h, there is a point $\xi_{h,\nu}$ in $[a, b]$ so that the sequence $u_h^{(\nu)}(\xi_{h,\nu})$, $h = 1, 2, \ldots$, is bounded. This is proven by first noting from (34b, c), (44) and (45a, b) that $\int_a^b (f_o + g_o) u_h^2 \, dx$ must be bounded. Hence, since $f_o + g_o \geq 0$, $\int_{x_1}^{x_2} (f_o + g_o) u_h^2 \, dx$ must be bounded, where (x_1, x_2) is any interval in $[a, b]$. From assumptions (44) there must exist an interval (x_1, x_2) in $[a, b]$ for which $f_o(x) + g_o(x) > 0$. Consequently, from the auxiliary theorem (j) of section 15, $\int_{x_1}^{x_2} u_h^2 \, dx$ will be bounded. Statement (1) then follows from the auxiliary theorem (f) with $p = m-1$, $q = 0$ there.

With the use of statement (1) it can be shown that: (2) For each ν, $0 \leq \nu \leq m-1$, the sequence $u_h^{(\nu)}(x)$ $(h = 1, 2, \ldots)$ is uniformly bounded in $[a, b]$. For,

$$u_h^{(m-1)}(x) = u_h^{(m-1)}(\xi_{h,m-1}) + \int_{\xi_{h,m-1}}^{x} u_h^{(m)} \, dx \tag{46}$$

But from (34b), (45b), and assumptions (44), the sequence $\int_a^b f_m (u_h^{(m)})^2 \, dx$ must be bounded; hence, since $f_m(x) > 0$ and is continuous in $[a, b]$, the sequence $\int_a^b (u_h^{(m)})^2 \, dx$ must also be bounded (auxiliary theorem (i)). Moreover, from (40b),

$$\int_a^b 1 \cdot |u_h^{(m)}| \, dx \leq \left[(b-a) \int_a^b (u_h^{(m)})^2 \, dx \right]^{1/2} \quad ;$$

hence $\int_a^b |u_h^{(m)}| \, dx$ must be bounded. It then follows from Eq. (46), in conjunction with statement (1), that the sequence $u_h^{(m-1)}(x)$ is uniformly bounded in $[a, b]$. From Eq. (46) with m successively now replaced by $(m-1)$, $(m-2)$, etc., statement (2) is readily seen to follow for each of the ν indicated.

It will now be proven that: (3) For each ν, $0 \leq \nu \leq m-1$, the sequence $u_h^{(\nu)}(x)$ is equicontinuous in $[a, b]$. For $0 \leq \nu \leq m-2$, statement (3) follows readily by observing, from statement (2) that for each such ν there will be a positive C_ν, independent of h, such that for any x_1 and x_2 in $[a, b]$, $|u_h^{(\nu)}(x_2) - u_h^{(\nu)}(x_1)| = \left| \int_{x_1}^{x_2} u_h^{(\nu+1)}(x)\, dx \right| \leq |x_2 - x_1| C_\nu$. Thus, taking $\delta(\epsilon) < \epsilon / C_\nu$, the equicontinuity definition (given in auxiliary theorem (m)) is seen to be satisfied for the sequence $u_h^{(\nu)}(x)$, $0 \leq \nu \leq m-2$. For $\nu = m-1$, it is first observed that since $f_m(x) > 0$ and is continuous in $[a, b]$, there is a positive D such that $f_m(x) \geq D$ in $[a, b]$. Hence, using also (40b),

$$\{u_h^{(m-1)}(x_2) - u_h^{(m-1)}(x_1)\}^2 = \left[\int_{x_1}^{x_2} 1 \cdot u_h^{(m)}(x)\, dx \right]^2$$

$$\leq |x_2 - x_1| \left| \int_{x_1}^{x_2} (u_h^{(m)})^2 dx \right| \leq (x_2 - x_1) \frac{1}{D} \int_a^b f_m (u_h^{(m)})^2 dx$$

$$\leq |x_2 - x_1| \overline{\Phi}(u_h)/D,$$ from (34b) and (44). Since by (45b), $\overline{\Phi}(u_h)$ is bounded, statement (3) is thus seen to hold for $\nu = m - 1$.

One can now prove the following: (4) For a suitably chosen subsequence (which may again be denoted by h) of the indices h, the functions $u_h(x)$ and their derivatives $u_h^{(\nu)}(x)$, $0 \leq \nu \leq m-1$, converge uniformly to an admissible function $U(x)$ and to $U^{(\nu)}(x)$, respectively, where $U(x)$ has continuous derivatives through order (m-1) in $[a, b]$, and satisfies the geometric boundary conditions. This statement, which is the main result of this Section, follows by first observing from statements (2) and (3), and the theorem of Arzela (auxiliary theorem (m)) applied for $\nu = m-1$, that a subsequence (which may still be denoted by h) of the indices h can be chosen so that

$$\lim_{h \to \infty} u_h^{(m-1)}(x) = U_{m-1}(x) \qquad (47)$$

where $U_{m-1}(x)$ is a continuous function in $[a, b]$. Moreover, from statement (2) applied to $u_h^{(\nu)}(a)$ and from auxiliary theorem (g) it follows that one can choose the above subsequence so that each of the sequences $u_h^{(\nu)}(a)$, $0 \leq \nu \leq m-1$, converges. From auxiliary theorem (i) with $f_j(x) = u_h^{(m-2)}(x)$, and $\xi = a$, and from the uniform convergence in (47), it then follows that the sequence $u_h^{(m-2)}(x)$ is uniformly convergent in $[a, b]$, with a limit function $U_{m-2}(x)$ such that $U'_{m-2}(x) = U_{m-1}(x)$. Application of auxiliary theorem (i) now to the sequence $u_h^{(m-3)}(x)$ shows that the sequence $u_h^{(m-3)}(x)$ will converge uniformly to a function $U_{m-3}(x)$, where $U'_{m-3}(x) = U_{m-2}(x)$. Further successive application of auxiliary theorem (i) thus shows that $u_h^{(\nu)}(x)$ converges uniformly to a limit function $U_\nu(x)$, where for $0 \leq \nu \leq m-2$, $U'_\nu(x) = U_{\nu+1}(x)$. Denoting $U_0(x)$ by $U(x)$, it is seen that statement (4) will readily follow. The fact that $U(x)$ will satisfy the geometric boundary conditions (which involve only $U^{(\nu)}$, $0 \leq \nu \leq m-1$) follows, of course, from the rest of statement (4) just proven, and from the fact that each $u_h(x)$ satisfies these conditions. Statement (4) and Eq. (45a) imply that

$$\overline{\mathfrak{J}}(U) = 1 \tag{48}$$

19. Differentiability of U(x) through 2m'th order. The function $U(x)$ obtained in Section 18 will now be shown to have continuous derivatives in $[a, b]$ through order $2m$. Differentiability of $U(x)$ beyond the $(m-1)$th order must be proved for at least two reasons. First, in order to show that $\overline{R}(U) = B$, it will be necessary to show that the derivative of at least m'th order of U exists in $[a, b]$. Second, it will then be desired to apply the results of Section 13 to show that U must be a comparison function minimizing the Rayleigh quotient R. The analysis of Section 13, however, assumes the 2m'th differentiability of U.

The following statement (A) will first be proven.

A. Let $v_h(x)$ be a sequence of admissible functions for which $\overline{\Phi}(v_h)$ and $\overline{\Psi}(v_h)$ are bounded, and for which

$$\overline{\Psi}(u_h, v_h) = 0 \tag{49a}$$

Then

$$\lim_{h \to \infty} \overline{\Phi}(u_h, v_h) = 0 \tag{49b}$$

Proof: It is noted (Eqs. (34f)) that for admissible functions u, v and self-adjoint boundary conditions, $\overline{\Phi}(u, v) = \overline{\Phi}(v, u)$, $\overline{\Psi}(u, v) = \overline{\Psi}(v, u)$. Hence Eq. (39) will hold for $\overline{\Phi}$ and $\overline{\Psi}$. Let now r_h and s_h be any chosen sequence of constants. Then $w_h = r_h u_h + s_h v_h$ will be admissible functions, and hence $\overline{\Phi}(w_h)/\overline{\Psi}(w_h) \geq B$. From Eqs. (39) for $\overline{\Phi}$ and $\overline{\Psi}$, (45a), and (49a), this inequality, letting $s_h > 0$, is found to imply

$$2\left(\frac{r_h}{s_h}\right)\overline{\Phi}(u_h, v_h) + \left(\frac{r_h}{s_h}\right)^2\left[\overline{\Phi}(u_h) - B\right] \geq -\left[\overline{\Phi}(v_h) - B\,\overline{\Psi}(v_h)\right] \tag{49c}$$

This must hold for all h, r_h and s_h. It is now observed that by the hypothesis, the entire quantity on the right side of (49c) will remain bounded as $h \to \infty$, and be negative or zero. Moreover, although $[\overline{\Phi}(u_h) - B]$ will be non-negative, it will approach zero as $h \to \infty$ (Eq. (45b)). Consequently, in order that (49c) hold as $h \to \infty$ for arbitrary (r_h/s_h), Eq. (49b) must hold. For otherwise, for some sufficiently large h, one could always find an (r_h/s_h) of opposite sign to $\overline{\Phi}(u_h, v_h)$ in such a way that (49c) would be violated.

With the aid of statement (A), the following will be proven:

(B) For every admissible function w(x),

$$\overline{\Phi}(u_h, w) - B\,\overline{\Psi}(u_h, w) \to 0 \quad \text{as } h \to \infty. \tag{49d}$$

To prove this, let

$$v_h = w - u_h\,\overline{\Psi}(u_h, w) \tag{49e}$$

Then v_h will be an admissible function; Eq. (39) may therefore be applied to $\bar{\Phi}$, to yield

$$\bar{\Phi}(v_h) = \bar{\Phi}(w) - 2\bar{\Psi}(u_h, w)\,\bar{\Phi}(u_h, w) + \bar{\Psi}^2(u_h, w)\,\bar{\Phi}(u_h) \tag{49f}$$

From (45a, b), $\bar{\Phi}(u_h)$ and $\bar{\Psi}(u_h)$ will be bounded as $h \to \infty$; hence from (49a), $\bar{\Phi}(u_h, w)$ and $\bar{\Psi}(u_h, w)$ will be bounded as $h \to \infty$. Therefore (49f) implies that also $\bar{\Phi}(v_h)$ will be bounded. From (39) applied to $\bar{\Psi}(v_h)$, and from (45a), it follows that $\bar{\Psi}(v_h) = \bar{\Psi}(w) - \bar{\Psi}^2(u_h, w)$. Hence the $\bar{\Psi}(v_h)$ will be bounded. It is finally noted from (49e) that $\bar{\Psi}(u_h, v_h) = \bar{\Psi}(u_h, w) - \bar{\Psi}(u_h, w)\cdot \bar{\Psi}(u_h) = 0$. Therefore statement (A) can be applied here, to yield $\bar{\Phi}(u_h, v_h) \to 0$ as $h \to \infty$. With v_h given by (49e), this yields $\bar{\Phi}(u_h, w) - \bar{\Phi}(u_h)\,\bar{\Psi}(u_h, w) \to 0$, which in view of (45b) yields (49d).

It can now be shown that U(x) has a continuous 2m'th derivative in [a, b]. Let w(x), as in the auxiliary theorem (k) of Section 15, be an arbitrary function with a continuous (m + 1)th derivative in [a, b], satisfying the boundary conditions (41a). For such a function $\bar{M}_o(u_h, w) = 0$, $\bar{N}_o(u_h, w) = 0$, and

$$\bar{\Phi}(u_h, w) - B\bar{\Psi}(u_h, w) = \int_a^b \Big(\sum_{\nu=0}^{m} f_\nu\, u_h^{(\nu)}\, w^{(\nu)}\Big)dx - B\int_a^b \Big(\sum_{\nu=0}^{n} g_\nu\, u_h^{(\nu)}\, w^{(\nu)}\Big)dx \tag{49g}$$

(cf. Eq. (34e)). By integration by parts, each term on the right side of (49g) can be transformed so that it will have a factor of $w^{(m+1)}$. For this purpose, as in Ref. 13, let

$$\left.\begin{aligned} f_\nu u_h^{(\nu)} &\equiv F_\nu^{[0]}(u_h)\,; \quad & g_\nu u_h^{(\nu)} &\equiv G_\nu^{[0]}(u_h) \\ \int_a^x F_\nu^{[k-1]}(u_h)\,dx &\equiv F_\nu^{[k]}(u_h)\,; \quad & \int_a^x G_\nu^{[k-1]}(u_h)\,dx &\equiv G_\nu^{[k]}(u_h) \end{aligned}\right\} \tag{49h}$$

Note that for $k > 0$ the $F_\nu^{[k]}(u_h)$ are essentially integrals (k = 1) or iterated

integrals ($k > 1$) of $f_\nu u_n^{(\nu)}$. Integrating by parts, and using conditions (41a), one then finds

$$\int_a^b f_\nu u_h^{(\nu)} w^{(\nu)} dx = (-1)^{m+1} \int_a^b F_\nu^{[m-\nu+1]} (u_h) w^{(m+1)} dx \tag{49i}$$

and a similar equation with f and F replaced by g and G, respectively.

From statement (B) (see Eq. (49d)), the left side of (49g) approaches zero as $h \to \infty$, and hence from (49i),

$$\lim_{h\to\infty} \int_a^b \left\{ \sum_{\nu=0}^{m} F_\nu^{[m-\nu+1]} (u_h) - B \sum_{\nu=0}^{n} G_\nu^{[m-\nu+1]} (u_h) \right\} w^{(m+1)} dx = 0 \tag{49j}$$

For the term with $\nu = m$, it is noted that

$$F_m^{[1]} (u_h) \equiv \int_a^x f_m u_h^{(m)} dx = f_m u_h^{(m-1)} \Big|_a^x - \int_a^x f_m' u_h^{(m-1)} dx \tag{49k}$$

Consequently, it is seen that only u_h and its derivatives through (m-1)th order appear in (49j). Hence in view of statement (4) of Section 18, one may replace u_h by U in (49j) in passing to the limit, and infer

$$\int_a^b \left\{ \sum_{\nu=0}^{m} F_\nu^{[m-\nu+1]} (U) - B \sum_{\nu=0}^{n} G_\nu^{[m-\nu+1]} (U) \right\} w^{(m+1)} dx = 0$$

Applying the auxiliary theorem (k) of Section 15, it now follows that

$$\sum_{\nu=0}^{m} F_\nu^{[m-\nu+1]} (U) - B \sum_{\nu=0}^{n} G_\nu^{[m-\nu+1]} (U) = P_m(x) \tag{49ℓ}$$

where $P_m(x)$ is a polynomial of degree m or less. Hence the left side of (49ℓ) is (continuously) differentiable m (and more) times.

Recalling from statement (4) of Section 18 that U is (continuously) differentiable at least through order (m-1), it is noted that each individual term on the left side of (49ℓ) for which $0 \leq \nu \leq m-1$ is differentiable. This includes all terms in (49ℓ) except the single term $F_m^{[1]}(U)$, which must therefore also be differentiable. But by (49k), noting that its right

side involves only $u_h^{(m-1)}$, it follows that

$F_m^{[1]}(U) = f_m U^{(m-1)} \Big|_a^x - \int_a^x f'_m U^{(m-1)} dx$. Since the integral here is differentiable, it follows that $f_m U^{m-1}$, and hence that $U^{(m-1)}$, is differentiable. Thus U must have continuous derivatives in [a, b] through at least m' th order.

To prove U differentiable beyond the m' th order, it is noted, from the definitions (49h), that by differentiating (49*l*) now once, one obtains

$$f_m U^{(m)} + \sum_{\nu=0}^{m-1} F_\nu^{[m-\nu]}(U) - B \sum_{\nu=0}^{n} G_\nu^{[m-\nu]}(U) = P_{m-1} \tag{49m}$$

Each individual term beyond the first on the left side of Eq. (49m) is differentiable. Hence the first term must also be differentiable, and this readily implies $U^{(m)}$ differentiable. By successively differentiating (49m) now one can similarly conclude successively that $U^{(m+1)}$, $U^{(m+2)}$, ..., are differentiable, until one finally concludes that U has continuous derivatives in [a, b] through order 2m.

20. Existence of minimizing functions for $\overline{R}$ and R. It will now be shown that

$$\overline{R}(U) = B \tag{50a}$$

Since U is admissible, it follows from (49d) that

$$\overline{\Phi}(u_h, U) - B\overline{\Psi}(u_h, U) \to 0 \text{ as } h \to \infty \tag{50b}$$

In passing to the limit here, statement (4) of Section 18 implies that one may replace u_h by U on the left side of (50b) in all terms other than the single term containing $u_h^{(m)}$. This term is

$$\int_a^b f_m u_h^{(m)} U^{(m)} dx = u_h^{(m-1)} f_m U^{(m)} \Big|_a^b - \int_a^b u_h^{(m-1)}(f_m U^{(m)})' dx \tag{50c}$$

One may now consider the limit of the right side of (50c) as $h \to \infty$, and note that here one may replace u_h by U in the limit. The result then becomes identically $\int_a^b f_m U^{(m)} U^{(m)} dx$, which is equivalent to replacing u_h by U on the left side of (50c). It follows then that (50b) implies $\overline{\Phi}(U) - B\overline{\Psi}(U) = 0$, which is equivalent to (50a), since $\overline{\Psi}(U) = 1$.

Thus, it has been proved that the function U(x) is indeed a minimizing admissible function for $\overline{R}(u)$. It has also been proved that this function has continuous derivatives through order 2m in [a, b]. Hence from Section 13, it follows that U will also be a comparison function satisfying (1)-(3), and hence an eigenfunction of (1)-(3) corresponding to the lowest eigenvalue $\lambda_1 = B$. Thus U is also a comparison function minimizing the Rayleigh quotient R. The derivation given here shows that when the energy quotient $\overline{R}$ is minimized, this eigenfunction can be obtained from a minimizing sequence of admissible functions which need only satisfy the geometric boundary conditions, and have a continuous (m - 1)th, and piecewise continous m'th derivative in [a, b]. The existence of relevant minimizing functions for $\overline{R}$ (and thence for R) for the k'th mode can be similarly proven, with the functions $u_h(x)$ now required to be orthogonal, in the sense of $\overline{\Psi}(u_h, y_i)$, to the first (k - 1) eigenfunctions.

It is noted, finally, that in assumptions (44) the condition $f_o + g_o \not\equiv 0$, which is used only in the proof of statement (1) of Section 18, can, if necessary, often be relaxed. It will suffice to illustrate this with a differential equation of the form $(p(x)\, y'')'' + \lambda y'' = 0$, $p(x) > 0$. Such an equation pertains, for example, to the static buckling of a column. Clearly, the condition $f_o + g_o \not\equiv 0$ does not hold here. It is noted, however, from the remainder of assumptions (44) and from Eqs. (34b, c) and (45a, b) that in general, regardless of f_o and g_o, $\int_a^b g_1(x)\, u_h'^2 dx$ must be bounded as $h \to \infty$. In the

present example, $g_1(x) = 1$; hence $\int_a^b u'^2_h \, dx$ must be bounded. Therefore, from auxiliary theorem (ℓ) of Section 15 with $q = 1$, statement (1) of Section 18 remains valid at least for $1 \leq \nu \leq m - 1$. If now (as would usually be the case in the present illustration) one of the boundary conditions requires that y itself vanish at $x = a$ or b, then statement (1) will hold also for $\nu = 0$, and thus for all $0 \leq \nu \leq m - 1$, as previously. If, on the other hand, y is not prescribed to vanish at an end, but is included in the boundary conditions and appears in $\overline{M}_o$ or $\overline{N}_o$, then from the assumptions $\overline{M}_o \geq 0$, $\overline{N}_o \geq 0$ and the other assumptions in (44), in conjunction with (45a, b), it may be readily inferred that y itself must remain bounded at an end point, so that statement (1) is valid also for $\nu = 0$. Finally, if the boundary conditions do not involve y at all, then for self-adjointness, y will not appear in either $\overline{M}_o$ or $\overline{N}_o$ (for admissible functions), and y itself will hence not be relevant. The existence proof can then be carried out for $\nu \geq 1$, to show that there will exist an admissible function $U'(x)$ for which $\overline{R}$ attains its greatest lower bound (minimum).

21. Unbounded growth of the eigenvalues. When sufficient conditions such as (44) hold so that the minimizing functions for R or $\overline{R}$, appearing in the theorems of Sections 8, 9 and 13 exist, then as seen in Section 9, there will be an infinite, but discrete, set of eigenvalues λ_k of (1)-(3) which can be ordered in a non-decreasing sequence $\lambda_1 \leq \lambda_2 \leq \ldots$. Henceforth under such conditions, the system (1)-(3) will be called "discrete". It will now be shown that for such systems the eigenvalues are unbounded as $k \to \infty$, i.e.

$$\lim_{k \to \infty} \lambda_k = \infty \tag{51}$$

This can be proven by supposing the contrary to hold. Then the

λ_k would have an upper bound, and from auxiliary theorem (e) of Section 15,

$$\lim_{k \to \infty} \lambda_k = \Lambda \tag{52a}$$

where Λ is a finite number. Let y_k be the corresponding normalized eigenfunctions. Then, since $\overline{R}(y_k) = \lambda_k$,

$$\overline{\Psi}(y_k) = 1, \qquad \lim_{k \to \infty} \overline{\Phi}(y_k) = \Lambda \tag{52b}$$

Thus the y_k would satisfy the same conditions, namely admissibility and Eqs. (45a, b) (with B replaced by Λ) as the u_h functions of Section 18. Following now the proof (statements (1)-(4)) in that Section, it is then clear (under assumptions (44)) that from the y_k one could form a subsequence of functions, that may still be denoted by y_k, which would converge uniformly in [a, b] to an admissible function Y (say); moreover the $y_k^{(\nu)}$ would converge uniformly to $Y^{(\nu)}$, $0 \leq \nu \leq m - 1$. From this it follows that $y_p^{(\nu)}(x) - y_q^{(\nu)}(x) \to 0$ in [a, b] as $p, q \to \infty$; hence, since $n \leq m-1$,

$$\lim_{p, q \to \infty} \overline{\Psi}(y_p - y_q) = 0 \tag{52c}$$

Since Eq. (39) holds for $\overline{\Psi}$, however,

$$\overline{\Psi}(y_p - y_q) = \overline{\Psi}(y_p) - 2\overline{\Psi}(y_p, y_q) + \overline{\Psi}(y_q) \tag{52d}$$

Hence, since $\overline{\Psi}(y_p) = \overline{\Psi}(y_q) = 1$ and $\overline{\Psi}(y_p, y_q) = 0$, it follows from (52d) that in general, for any two orthonormal eigenfunctions y_p and y_q,

$$\overline{\Psi}(y_p - v_q) = 2 \tag{52e}$$

This contradicts (52c), and hence contradicts the assumption that (51) is false. Thus (51) is proven. It may be noted that (51) implies that any repeated eigenvalues can each have only a finite multiplicity.

V. COMPLETENESS, CLOSEDNESS AND EXPANSION THEOREMS

22. Bessel's Inequality and Parseval's Equation. Let A be a positive definite linear differential self-adjoint operator, and let $\varphi_1, \varphi_2, \ldots$ be a set of comparison orthonormal functions with respect to A, i.e., satisfying

$$(A\varphi_i, \varphi_j) = (\varphi_j, A\varphi_i) = \delta_{ij} \tag{53a}$$

Then the series $\sum_{i=1}^{\infty} c_i \varphi_i(x)$ will be said to converge, in the norm of the operator A, to a function f(x) in [a, b] if

$$\lim_{p \to \infty} \| f - u_p \|_A^2 \equiv \lim_{p \to \infty} \int_a^b (f - u_p) A (f - u_p)\, dx = 0 \tag{53b}$$

where

$$u_p(x) = \sum_{i=1}^{p} c_i \varphi_i(x) \tag{53c}$$

The quantity $e_p^2 \equiv \| f - u_p \|_A^2$ may be regarded as the mean squared error in [a, b], with the operator A, of the finite series representation, $\sum_{i=1}^{p} c_i \varphi_i$, of f(x). When A = I, the identity operator, $|e_p|$ is the "norm" of $(f - u_p)$, and the subscript I (in place of A) is then omitted in the notation (cf. Section 2). Moreover, one then simply speaks of "convergence in the mean" of u_p to f, without mention of an operator. For the more general operator A, the terminology[8] "energy convergence" may be used to mean that (53b) is satisfied; $|e_p|$ may then be called the "energy norm", or "A-norm" of $(f - u_p)$.

A further condition which will be imposed on the operator A here is that it be "positive-bounded-below"[8], also called more simply "strictly positive". This means that if u is any comparison function, then

$$||u||_A \geq \gamma\, ||u|| \tag{53d}$$

where γ is some fixed positive constant, and where, in accordance with the notation herein, (see also Section 2), $||u||_A \equiv (Au, u)^{1/2}$, $||u|| \equiv (u, u)^{1/2}$. The following interpretation of (53d) is particularly pertinent here. Consider the eigenvalue problem $Ay = \lambda y$, with associated boundary conditions; A is assumed self-adjoint and positive definite. Suppose, in addition, that the spectrum of eigenvalues is discrete. Then there will be a lowest eigenvalue $\lambda_1 > 0$, which will be the minimum value of the Rayleigh quotient $R(u) \equiv (Au, u)/(u, u) \equiv ||u||_A^2 / ||u||^2$. Hence $||u||_A^2 / ||u||^2 \geq \lambda_1$, whence (51d) holds with $\gamma = \lambda_1^{1/2}$. Thus, if A is positive definite, self-adjoint, and the eigenvalue problem $Ay = \lambda y$ has a discrete spectrum, then (53d) will hold. An important consequence of (53d) is that under this condition, convergence in the norm of A will imply convergence in the mean. This follows readily by noting that if f is a comparison function, then so is $f - u_p$; hence from (53d), $0 \leq ||f - u_p|| \leq (1/\gamma)||f - u_p||_A$. Therefore $||f - u_p||_A \rightarrow 0$ implies $||f - u_p|| \rightarrow 0$ as $p \rightarrow \infty$.

If, for all f of a given class, a set of constants c_i (depending on f) exists so that (53b) holds, then the set of functions φ_i is said to be complete, in the norm of A, for such functions. By formally writing $f = \sum c_j \varphi_j$, operating with A on both sides term by term, multiplying by φ_i, integrating, noting Eqs. (53a) and assuming also $(\varphi_i, Af) = (f, A\varphi_i)$, one obtains

$$c_i = \int_a^b f A(\varphi_i)\, dx \equiv (f, A\varphi_i) \tag{54}$$

The quantities $(f, A\varphi_i)$ are called the (generalized) Fourier coefficients of the function f. Denoting $\alpha_i \equiv (f, A\varphi_i)$, it is readily found that for <u>any</u> set of c_i,

$$e_p^2 = (f,\ Af) - 2 \sum_{i=1}^{p} c_i \alpha_i + \sum_{i=1}^{p} c_i^2$$

$$= (f,\ Af) - \sum_{i=1}^{p} \alpha_i^2 + \sum_{i=1}^{p} (c_i - \alpha_i)^2 \tag{55a}$$

From (55a) it follows that the mean squared error will be a minimum with respect to the c_i (for given f and φ_i) when $c_i = \alpha_i$. Thus the Fourier coefficients are the values of the c_i for which e_p^2 is minimized. Moreover, when c_i is given by Eq. (54), (i.e., $c_i = \alpha_i$), it follows from (55a) that

$$\sum_{i=1}^{p} c_i^2 = (f,\ Af) - e_p^2 \tag{55b}$$

Since $e_p^2 \geq 0$ for any p, it follows that

$$\sum_{i=1}^{\infty} c_i^2 \leq (f,\ Af) \tag{56}$$

(56) is known as Bessel's inequality, and shows that in general, if (f, Af) is bounded and the c_i are the Fourier coefficients of f, then $\sum_{i=1}^{\infty} c_i^2$ converges. Moreover, from (55b) it follows that $\lim_{p\to\infty} e_p^2 = 0$ if and only if

$$\sum_{i=1}^{\infty} c_i^2 = (f,\ Af) \tag{57}$$

Eq. (57) is known as Parseval's equality, or the completeness relation. It is seen to be equivalent to the completeness, in the A-norm, of the set of functions φ_i.

The Parseval equation (57) can be generalized by considering any two comparison functions u(x) and v(x) such that the completeness relation (57) holds for u and v separately, and also for (u + v). Then inserting f = (u + v) in (57), and letting a_i and b_i denote the Fourier coefficients (defined by (54)) of u and v, respectively, Eq. (57) yields

$$\sum_{i=1}^{\infty} (a_i + b_i)^2 = \int_a^b (u + v)\, A\, (u + v)\, dx$$

$$= (u, Au) + (v, Av) + 2\,(u,\ Av)$$

Observing, from (57), that $\sum_{i=1}^{\infty} a_i^2 = (u,\ Au)$ and $\sum_{i=1}^{\infty} b_i^2 = (v, Av)$, it follows that

$$\sum_{i=1}^{\infty} a_i\, b_i = (u,\ Av) \tag{58}$$

Eq. (58) may be regarded as the "generalized" Parseval equation.

23. Completeness of the eigenfunctions in the norms of N and of M. Consider now the eigenvalue problem (1) - (3), with M and N self-adjoint and positive definite. Further conditions on $f_\nu(x)$ and $g_\nu(x)$, such as assumptions (44), are assumed which are sufficient for the existence of the minima of the Rayleigh quotient with respect to comparison functions. Then it will be shown first that, in the norm of the operator N, the orthonormalized eigenfunctions $y_i(x)$ (satisfying (57a) with A = N; cf. Eqs. (8a)) form a complete set for comparison functions f(x), i.e.

$$\sum_{i=1}^{\infty} c_i^2 = \int_a^b f\, N(f)\, dx \tag{59}$$

where, in accordance with (54),

$$c_i = \int_a^b f(x)\, N(y_i)\, dx \tag{60}$$

To prove Eq. (59), let

$$r_p(x) = f(x) - \sum_{i=1}^{p} c_i\, y_i(x) \tag{61a}$$

where c_i is given by Eq. (60). Then $r_p(x)$ is a comparison function, and moreover,

$$\int_a^b r_p N(y_i)\, dx = 0 \qquad (i = 1, \ldots, p) \tag{61b}$$

Thus r_p is orthogonal to the first p eigenfunctions. Consequently, from the recursive minimum characterization of λ_{p+1},

$$\Phi(r_p)/\psi(r_p) \equiv R(r_p) \geq \lambda_{p+1} \tag{61c}$$

From $M(y_i) = \lambda_i N(y_i)$, Eq. (61b) implies

$$\int_a^b r_p M(y_i)\, dx = 0 \qquad (i = 1, \ldots, p) \tag{61d}$$

Putting $f = r_p + \sum_{i=1}^{p} c_i y_i(x)$, and using (8b), (10) and (61d), one readily finds

$$\Phi(f) \equiv \int_a^b f\, M(f)\, dx = \Phi(r_p) + \sum_{i=1}^{p} c_i^2 \lambda_i \tag{61e}$$

Since $\Phi(f)$ is fixed and bounded, and $\Phi(f) > 0$, $\Phi(r_p) \geq 0$, $\lambda_i > 0$ it follows from (61e) that $\Phi(r_p)$ is bounded as $p \to \infty$. But from (61c) $\psi(r_p) \leq \Phi(r_p)/\lambda_{p+1}$. Since $\lambda_p \to \infty$ as $p \to \infty$, this then implies

$$\lim_{p \to \infty} \psi(r_p) = 0 \tag{62}$$

Eq. (62) is identical to (53b), with A replaced by N. Thus the completeness, in the norm of N, of the set $y_i(x)$ for comparison functions f is proven. Eq. (59) then follows from Parseval's equality (57) with A = N.

From Eq. (61e), since $\Phi(r_p) \geq 0$, it follows that

$$\sum_{i=1}^{\infty} c_i^2 \lambda_i \leq \int_a^b f\, M(f)\, dx \tag{63}$$

Inequality (63) shows that $\sum_{i=1}^{\infty} c_i^2 \lambda_i$ is convergent.

The above proof can be modified by using the energy quotient $\overline{R}$, and the associated functions $\overline{\Phi}$ and $\overline{\psi}$ in place of R, Φ and ψ (cf. Section 13).

The completeness relation then becomes

$$\sum_{i=1}^{\infty} c_i^2 = \bar{\Psi}(f) \tag{64a}$$

where now

$$c_i = \bar{\Psi}(f, y_i) \tag{64b}$$

Moreover, (64a, b) now hold for admissible functions f which need only satisfy the geometric boundary conditions, and have a continuous (m-1)th, and piecewise continuous m'th, derivative in [a, b].

The completeness relations derived above can be extended to a wider class of functions f than described there. For this purpose, it is first noted, as observed in Refs. 4 and 6, that for the special case of $N(y) \equiv g_0(x)\, y$, the set of comparison functions, to be denoted by $\bar{f}$, for the problem (1) - (3) is "dense" in the space of square-integrable functions f, that is: given such an f and any $\epsilon > 0$, there exists an $\bar{f}$ such that

$$||f - \bar{f}||_N < \epsilon \tag{65a}$$

For more general N(y) as defined by Eq. (3) one may consider the Dirichlet form (cf. Eq. (28)), obtained by integrations by parts, of $||f - \bar{f}||_N \equiv \int_a^b (f - \bar{f})\, N\, (f - \bar{f})\, dx \equiv \psi(f - \bar{f})$. This will indicate that the denseness property (65a) of the set of comparison functions $\bar{f}$ will hold if f is now a function whose N-norm exists, and which satisfies enough of the boundary conditions (2) so that the boundary terms in the Dirichlet form of $\psi(f - \bar{f})$ can be made zero or as small as desired. The derivatives involved here are of order (2n-1) and less, and examination of $\psi(f - \bar{f})$ will indicate that the above requirement on the boundary terms can be satisfied by prescribing 2n or fewer boundary conditions for f. By the use of Eq. (65a) for such functions f, it can be shown, following the type of proof in Refs. 4 or 6,

that the eigenfunctions of (1) - (3) will be complete in the norm of N for this more general class of functions. This implies that Eq. (59) will hold for such f; moreover, Eq. (58) will also hold, with A = N, and u and v any two functions of this class f.

With the use of the observations in the preceding paragraph, it can now be shown that the orthonormalized eigenfunctions y_i of (1) - (3) will be a complete set, in the norm of M, for comparison functions u. This can be proven by applying the generalized Parseval Eq. (58) with A = N, u = any given comparison function, and v a function such that

$$N(v) = M(u) \tag{65b}$$

and satisfying sufficient boundary conditions (as explained in the preceding paragraph) so that a comparison function $\bar{f}$ will always exist such that (65a) will hold with f = v, and so that N will be self-adjoint for such functions v. Then, in Eq. (58),

$$a_i = \int_a^b u\, N(y_i)\, dx \equiv c_i \tag{65c}$$

$$b_i = \int_a^b v\, N(y_i)\, dx = \int_a^b y_i\, N(v)\, dx = \int_a^b y_i\, M(u)\, dx$$
$$= \int_a^b u\, M(y_i)\, dx = \int_a^b u\, \lambda_i\, N(y_i)\, dx = \lambda_i\, c_i \tag{65d}$$

Eq. (58), in conjunction with Eqs. (65b) - (65d), then yields

$$\sum_{i=1}^{\infty} c_i^2\, \lambda_i = \int_a^b u\, M(u)\, dx \tag{66}$$

Eq. (66) is the same as Eq. (63), with the inequality sign now replaced by an equality. Eq. (66) signifies the completeness of the eigenfunctions,

in the norm of M, for comparison functions u^*.

24. Closedness theorems. An interesting and instructive alternative method of proving the completeness of the eigenfunctions, both in the norm of N and of M, will now be shown. For this purpose, the concept of closedness of a set of orthonormal functions φ_i will be introduced. This, as will be seen, is intimately related to completeness**. A set of orthonormal functions $\varphi_i(x)$ will be called closed with respect to functions f of a given class if, under a given positive definite operator A (assumed also to be linear and self-adjoint), there is no normalized function f which is orthogonal to every φ_i in the set. One might expect intuitively that closedness and completeness of a set are equivalent. This, indeed, is the case for functions f for which (f, Af) exists (in the generalized, Lebesgue sense). To prove that completeness implies closedness is comparatively simple. For, suppose a set is complete, but not closed. Then there would exist a normalized function f such that $c_i \equiv (f, A\varphi_i) = 0$ for all i. Hence $\lim_{p \to \infty} ||f - \sum_{i=1}^{p} c_i \varphi_i||_A = ||f||_A \equiv (f, Af) = 1$, which contradicts the supposition that the set φ_i is complete. The converse, namely that closedness implies completeness, can be proven by using the well-known Fischer-Riesz theorem of functional analys... This theorem states that

* Actually, as seen in this derivation, it is sufficient that u satisfy enough of the boundary conditions in (2) so that (65a) will hold, and so that $\int_a^b y_i \, M(u) \, dx = \int_a^b u \, M(y_i) \, dx$.

** In fact, the definitions of "completeness" and "closedness" of a set of functions are sometimes interchanged in the literature.

every Cauchy sequence*, f_n, of functions f for which the A-norm (f, Af) exists, converges in the norm of A to such a function. A proof of this theorem may be found, e.g., in Refs. 7 and 17 for $A = I$, the identity operator. The extension in the form stated above, to a general positive definite, self-adjoint operator A, satisfying also (53d), is proven by Mikhlin[8] (pp. 100-109). With this theorem, the statement that closedness implies completeness can be proven as follows. Suppose the set of orthonormal functions φ_i is not complete. Then by (56) there will be a function f(x) for which

$$(f, Af) - \sum_{i=1}^{\infty} c_i^2 > 0 \tag{67a}$$

where c_i is given by (54). Consider now the sequence of functions $g_p = f - \sum_{i=1}^{p} c_i \varphi_i$. Then $||g_p - g_q||_A^2 = \sum_{i=p+1}^{q} c_i^2 \to 0$ as $p, q \to \infty$, since $\sum_{i=1}^{\infty} c_i^2$ is convergent. Hence g_p is a Cauchy sequence, and by the Fischer-Riesz theorem must therefore converge, in the norm of A, to a function g(x) whose A-norm exists. This function will be orthogonal to all of the φ_i. For, $(g, A\varphi_i) = (g - g_p, A\varphi_i) + (g_p, A\varphi_i)$ for any p. But for $p \geq i$, $(g_p, A\varphi_i) = c_i - c_i = 0$. $\therefore$ $(g, A\varphi_i) = (g - g_p, A\varphi_i)$ for all $p \geq i$. Moreover, given any $\epsilon > 0$, there will be a P such that for $p > P$, $(g - g_p, A\varphi_i) < \epsilon$. Since $(g, A\varphi_i)$ is independent of p, it then follows that $(g, A\varphi_i) = 0$. It will now be shown, further, that $||g||_A^2 > 0$. For, noting that $g = \eta_p + g_p$, where $\eta_p \equiv g - g_p$, it is seen that

$$||g||_A^2 = ||\eta_p||_A^2 + 2(\eta_p, A g_p) + ||g_p||_A^2 \tag{67b}$$

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* i.e., a sequence in which $||f_p - f_q||_A^2 \to 0$ as $p, q \to \infty$.

This holds for all p; but for sufficiently large p the first three terms will be arbitrarily small in absolute value, while $||\varepsilon_p||_A^2 = (f, Af) - \sum_{i=1}^{p} c_i^2$. In view of (67a), it therefore follows that for sufficiently large p, the right side of (67b) will be positive. Hence $||g||_A^2 > 0$. Thus g is a normalizable function which is orthogonal to all the φ_i. Hence the φ_i cannot be closed, if they are not complete.

Suppose now that the system (1) - (3) is self-adjoint, with M positive semidefinite and N positive definite, and that the system has a discrete spectrum. Then it will be shown that the eigenfunctions of the problem (1) - (3) are closed, under the operator N, for comparison functions f. To prove this it is noted that λ_p is the minimum value of R(u) with respect to comparison functions u for which $\psi(u) = 1$ and

$$(u, Ny_1) = 0, \quad (u, Ny_2) = 0, \quad \ldots, \quad (u, Ny_{p-1}) = 0 \qquad (67c)$$

Thus, $\lambda_p \geq \lambda_{p-1} \geq \ldots \geq \lambda_1$. Suppose the set of eigenfunctions y_i were not closed. Then there would exist normalized comparison functions u which satisfy (67c) for all the y_i, i = 1, 2, ... For such functions u, R(u) must exist, i.e. be finite. Moreover, since $R(u) \geq 0$, R(u) must have a greatest lower bound, say $\bar{\lambda}$. But then $\bar{\lambda}$ must be at least as large as each of the λ_i, i = 1, 2, ... This, however, is impossible, since $\lambda_i \to \infty$ as $i \to \infty$. Hence the system of y_i must be closed. Note that this proof does not use any completeness theorems, and may hence be considered as an alternative proof of completeness.

In a similar fashion, it can be shown that the system of eigenfunctions y_i is closed not only under the operator N, but also under M, when M is positive-definite. To prove this, it is first recalled that the y_i are orthogonal with the operator M, i.e., $(y_i, M(y_j)) = 0$, $i \neq j$ (Eq. 8b).

It is then noted that the eigenvalue λ_p can be characterized as follows: λ_p is the minimum value of $\Phi(u)$ with respect to comparison functions u for which $\psi(u) = 1$ and $(u, M(y_i)) = 0$, $i = 1, 2, \ldots, p-1$. This can be shown in a manner quite similar to that already shown in Section 9 for the corresponding theorem involving N (instead of M): In the equation immediately before Eq. (27), the $\psi(\delta u, y_i)$ are replaced by $\Phi(\delta u, y_i)$, $i = 1, \ldots, k-1$. The proof then proceeds as in that section, with the observation that $N(y_i) = (1/\lambda_i) M(y_i)$, and with the result that $\kappa_i \Phi(y_i) = 2[1 - (\kappa_k/\lambda_i)] \Phi(u_k, y_i)$. But now $\Phi(u_k, y_i) = 0$. Hence, since $\Phi(y_i) > 0$, $\kappa_i = 0$ and the proof then continues exactly as in Section 9. With this characterization of the eigenvalues, the proof that the eigenfunctions are closed, under the operator M, is then the same as that in the preceding paragraph. From this one may now infer that the eigenfunctions Y_i, orthonormalized with the operator M, form a complete set for comparison functions f. Note that if $\psi(y_i) = 1$, then

$$Y_i(x) = y_i(x) \lambda_i^{-1/2} \tag{68a}$$

Moreover, in the series $\sum c_i' Y_i(x)$ to represent f now, the values of the c_i' according to Eq. (54), in conjunction with Eq. (60) for c_i, will be

$$\begin{aligned} c_i' &= \int_a^b f(x) M(Y_i) dx = \int_a^b f(x) \lambda_i N(Y_i) dx \\ &= \int_a^b f(x) \lambda_i N(y_i) \lambda_i^{-1/2} dx = \lambda_i^{1/2} c_i \end{aligned} \tag{68b}$$

Hence the series $\sum c_i' Y_i(x)$ becomes $\sum c_i y_i(x)$, as in the case of the operator N, and the completeness of the functions $Y_i(x)$ in the M-norm signifies that $\Phi(r_p) \to 0$ as $p \to \infty$, where r_p is still defined by Eq. (61a). Eq. (61e) then yields the M-norm completeness relation (66). This may be considered as an alternative proof of Eq. (66) to that given in Section 23.

25. Expansion theorems. Eqs. (59) and (66) are the chief results obtained in this Chapter, and, as will be seen in a second report, play an important role in the theory of energy methods. It is noted, for example, that Eq. (66) leads to the same result that would be obtained if one first formally writes $f(x) = \sum_{i=1}^{\infty} c_i\, y_i(x)$, operates termwise with M (which involves 2m differentiations) on both sides, multiplies both sides by f and then integrates termwise over [a, b]. Eq. (66) justifies mathematically the final result of such an operation, at least for any comparison function f. It should be observed, however, that in spite of the validity of the final results of such an operation, it has not yet been established here that the equation $f(x) = \sum_{i=1}^{\infty} c_i\, y_i(x)$ itself actually holds uniformly for all x in [a, b]. Such a statement would be an " expansion theorem". Expansion theorems are developed in Refs. 4 and 12 for Sturm-Liouville (second-order) equations, and in Refs. 1, 6 and 18 for more general systems of the type (1) - (3).

As will be seen in a subsequent report, it is actually the completeness relations of the type (59) and (66) which will suffice for the further developments to be given there of the theory of energy methods. The following remarks on expansion theorems may nevertheless be made here. Considering first A = I, the identity operator, in the definition (53b) of completeness, it is noted that convergence in the mean signifies that $\lim_{p \to \infty} [f(x) - \sum_{i=1}^{\infty} c_i\, \varphi_i\, (x)]$ will be zero "almost everywhere" (in the Lebesgue sense) in [a, b], i.e., the limit will be zero at all x in [a, b] with the possible exception of a set of points (of "zero measure") which

would not contribute to the integral in (53b)*.

For $A = I$, sufficient conditions for the uniform convergence of $\sum_{i=1}^{\infty} c_i \tau_i(x)$ to $f(x)$ in $[a, b]$ would be: $f(x)$ continuous, all $\tau_i(x)$ continuous, and $\sum_{i=1}^{\infty} c_i \tau_i(x)$ uniformly convergent in $[a, b]$.

Consider now the self-adjoint system (1) - (3) with M and N positive definite such that the eigenvalue spectrum is discrete, and such that the system

$$M(u) = \Lambda u \tag{69}$$

with the same boundary conditions will also have a discrete positive spectrum. Then M will be a strictly positive operator, i.e., M satisfies (53d) (with $A = M$, $\gamma = \Lambda_1 > 0$, the lowest eigenvalue of (69)). From the completeness, in the norm of M, of the eigenfunctions y_i of (1) - (3) for comparison functions f(x), established in Sections 23 and 24, it then follows that the y_i will also be complete in the sense of ordinary convergence in the mean (i.e. (53b) will hold with $A = I$, $\tau_i = y_i$). Consequently, the following statement holds: If $\sum_{i=1}^{\infty} c_i y_i(x)$ is uniformly convergent in $[a, b]$, where the y_i are the orthonormalized eigenfunctions of (1) - (3) with the operator N, and the c_i are given by (60), then $\sum_{i=1}^{\infty} c_i y_i(x)$ will converge uniformly to $f(x)$ for all x in $[a, b]$. It is shown in Ref. 1 (pp. 144-145) that actually $\sum_{i=1}^{\infty} c_i y_i^{(\nu)}(x)$, $0 \leq \nu \leq m - 1$, converges uniformly (and absolutely) in $[a, b]$. Consequently, the series $\sum_{i=1}^{\infty} c_i y_i(x)$ will converge uniformly to

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* As a simple example, with a sequence (instead of a series), let $f(x) \equiv 0$ in $[0, 1]$, and consider the sequence $f_p(x) = (1 + px)^{-1/2}$ in $[0, 1]$, $p = 1, 2, \ldots$. Then $\|f - f_p\|^2 = \|0 - f_p\|^2 = \int_0^1 (1 + px)^{-1} dx = (1/p) \log(1 + p)$. Hence as $p \to \infty$, $\|0 - f_p\|^2 \to 0$. Thus the sequence $f_p(x)$ converges in the mean to $f(x) \equiv 0$. The sequence $f_p(x)$ itself, however, is readily seen to converge to zero at all x in $[0, 1]$ except at $x = 0$, where $f_p = 1$ for all p (and hence $f_p \to 1$). Note that the sequence $f_p(x)$ does not converge uniformly in $(0, 1]$ to zero.

$f(x)$ for all x in $[a, b]$, and the series may, in fact, be differentiated termwise $(m - 1)$ times.

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